Two Dimensional Coordinate System and Graphs

**Cartesian Coordinate System**
- Vertical Axis or y-axis
- Horizontal axis or x-axis
- Origin
- Ordered pair
- x-coordinate or abscissa
- y-coordinate or ordinate
- quadrant

**Distance Formula:**
\[ d^2 = |x_2 - x_1|^2 + |Y_2 - Y_1|^2 \]
\[ d = \sqrt{(x_2 - x_1)^2 + (Y_2 - Y_1)^2} \]

**Midpoint Formula**

Midpoint of \( P_1P_2 \) is \( \left( \frac{x_1 + x_2}{2}, \frac{Y_2 + Y_1}{2} \right) \)
The graph of an equation in the two variables x and y is the set of all points whose coordinates satisfy the equation.

\[ y = 2x - 1 \]

<table>
<thead>
<tr>
<th>x</th>
<th>2x-1</th>
<th>y</th>
<th>(x, y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-2(-2)-1</td>
<td>-5</td>
<td>(-2, -5)</td>
</tr>
<tr>
<td>-1</td>
<td>-2(-1)-1</td>
<td>-3</td>
<td>(-1, -3)</td>
</tr>
<tr>
<td>0</td>
<td>-2(0)-1</td>
<td>-1</td>
<td>(0, -1)</td>
</tr>
<tr>
<td>1</td>
<td>2(1)-1</td>
<td>1</td>
<td>(1, 1)</td>
</tr>
<tr>
<td>2</td>
<td>2(2)-1</td>
<td>3</td>
<td>(2, 3)</td>
</tr>
</tbody>
</table>

\[ -x^2 + y = 1 \text{ or } y = x^2 + 1 \]

<table>
<thead>
<tr>
<th>x</th>
<th>x^2+1</th>
<th>y</th>
<th>(x, y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>(-2)^2+1</td>
<td>5</td>
<td>(-2, -5)</td>
</tr>
<tr>
<td>-1</td>
<td>(-1)^2+1</td>
<td>2</td>
<td>(-1, 1)</td>
</tr>
<tr>
<td>0</td>
<td>0^2+1</td>
<td>1</td>
<td>(0, 1)</td>
</tr>
<tr>
<td>1</td>
<td>1^2+1</td>
<td>2</td>
<td>(1, 2)</td>
</tr>
<tr>
<td>2</td>
<td>2^2+1</td>
<td>5</td>
<td>(2, 5)</td>
</tr>
</tbody>
</table>
\[ y = |x - 2| \]

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>5</td>
</tr>
<tr>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>-1</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>

**Definition of x-intercept and y-intercept:**

If \((x, 0)\) satisfies an equation, then point \((x, 0)\) is called an **x-intercept** of the graph of the equation.

If \((0, y)\) satisfies an equation, then point \((0, y)\) is called a **y-intercept** of the graph of the equation.
**Definition of a Circle:**
A circle is the set of points in a plane that are a fixed distance from a specified point. The distance is the **radius** of the circle, and the specified point is the **center** of a circle.

\[ (x-h)^2 + (y-k)^2 = r^2 \]

where \((h,k)\) is the center of the circle and \(r\) is the radius of the circle.
Example 1
Find equation of circle that has a center of C(-4,-2) and contains the point P(-1,2)

Example 2
Find the center and the radius of a circle that is given by

\[ x^2 + y^2 - 6x + 4y - 3 = 0 \]
A table, equation, or a graph represent a set of ordered pairs $(x, y)$ is called a relation.

**Definition of a Function**
A function is a set of ordered pairs in which no two ordered pairs have the same first coordinate.

**Example:**

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>-2</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
</tr>
</tbody>
</table>

Domain: Set of all first coordinates of ordered pair. (“Independent variable”)

Range: Set of all second coordinates of ordered pair. (“Dependent variable”)

![Graph of ordered pairs]
Function Notation:

\[ f(x) = x^2 + 1 \]

\[ f(2) = \]

\[ f(0) = \]

\[ f(a) = \]

\[ f(r+2) = \]

Piecewise – Defined Function:

\[ f(x) = \begin{cases} 
2x, & x < -2 \\
x^2, & -2 \leq x < 1 \\
4 - x, & x \geq 1 
\end{cases} \]

Definition of Increasing, Decreasing, Constant Function.

If \( a \) and \( b \) are elements of interval \( I \) that is a subset of domain of function \( f \), then

- \( f \) is increasing on \( I \) if \( f(a) < f(b) \) when \( a < b \)
- \( f \) is decreasing on \( I \) if \( f(a) > f(b) \) when \( a < b \)
- \( f \) is constant on \( I \) if \( f(a) = f(b) \) for all \( a \) and \( b \)
One to One Function:
A “one-to-one” function satisfies condition that no two ordered pairs have same second coordinate.

Use horizontal line test to check for a one-to-one function.

The Greatest integer Function. (Floor Function)

\[
\begin{align*}
\lfloor 2.7 \rfloor &= 2 \\
\lfloor -3.1 \rfloor &= -4 \\
\text{int}\left(\frac{7}{2}\right) &= 3
\end{align*}
\]

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-5 \leq x &lt; -4)</td>
<td>-5</td>
</tr>
<tr>
<td>(-4 \leq x &lt; -3)</td>
<td>-4</td>
</tr>
<tr>
<td>(-3 \leq x &lt; -2)</td>
<td>-3</td>
</tr>
<tr>
<td>(-2 \leq x &lt; -1)</td>
<td>-2</td>
</tr>
<tr>
<td>(-1 \leq x &lt; 0)</td>
<td>-1</td>
</tr>
<tr>
<td>(0 \leq x &lt; 1)</td>
<td>0</td>
</tr>
<tr>
<td>(1 \leq x &lt; 2)</td>
<td>1</td>
</tr>
<tr>
<td>(2 \leq x &lt; 3)</td>
<td>2</td>
</tr>
</tbody>
</table>

“Continuous function”
“Discontinuous function”
#75

\[
d_1 = 45 - 8t \\
\]

\[
d_2 = 6t \\
\]

\[
d = \sqrt{(5 - 8t)^2 + (6t)^2} \\
\]

#79

\[
\frac{15}{h} = \frac{3}{3 - r} \\
\]

\[
3h = \frac{15(3 - r)}{3} \\
h = 5(3 - r) \\
h = 15 - 5r \\
\]

#83

\[
d_2 = \sqrt{30^2 + x^2} \\
\]

\[
d_1 = \sqrt{20^2 + (40 - x)^2} \\
\]

\[
T(x) = \sqrt{900 + x^2} + \sqrt{400 + (40 - x)^2} \\
\]

<table>
<thead>
<tr>
<th>x</th>
<th>Total Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>74.72</td>
</tr>
<tr>
<td>10</td>
<td>67.68</td>
</tr>
<tr>
<td>20</td>
<td>64.34</td>
</tr>
<tr>
<td>30</td>
<td>64.79</td>
</tr>
<tr>
<td>40</td>
<td>70</td>
</tr>
</tbody>
</table>
## 2.3 Linear Functions

**Definition of a Linear Function:**
A function of the form \( f(x) = mx + b, m \neq 0 \) where \( m \) and \( b \) are real numbers, is a linear function of \( x \).

**Slope of a line \( m \):**
\[
m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}
\]
\[
m = \frac{\text{Rise}}{\text{Run}}
\]

The slope \( m \) of the line passing through the points \( P_1(x_1, y_1) \) and \( P_2(x_2, y_2) \) with \( x_1 \neq x_2 \) is given by
\[
m = \frac{y_2 - y_1}{x_2 - x_1}
\]
2.3

The graph of $x=a$ is a vertical line through $(a,0)$

The graph of $y=b$ is a horizontal line through $(0,b)$

**Slope-Intercept Form**

The graph of $fx=mx+b$ is a line with slope $m$ and $y$-intercept $(0,b)$

**Point-slope Form**

The graph of $y - y_1 = m(x - x_1)$ is a line with slope $m$ and passes through $(x_1, y_1)$

**Real solutions and x-Intercepts Theorem**

For every function $f$, the real number $c$ is a solution of $f(x)=0$ if and only if $(c,0)$ is an $x$-intercept of the graph $y=f(x)$
Parallel and Perpendicular Lines

Let $\ell_1$ be a graph of $f_1(x) = m_1x + b_1$

and $\ell_2$ be a graph of $f_2(x) = m_2x + b_2$

- $\ell_1$ and $\ell_2$ are parallel iff $m_1 = m_2$
- $\ell_1$ and $\ell_2$ are perpendicular iff $m_1 = -\frac{1}{m_2}$
Example 1  Find the slope of line through points (-5, -1) and (-3,4)

Example 2  Graph  $y = \frac{2}{3}x - 4$

Example 3  Graph  $3x - 2y = -6$

Example 4  Find equation of line in  $y = mx + b$  form through (-2, 3) and  $m = -4$

Method 1  

Method 2
Example 5  Find equation of line in $y=mx+b$ form through $(4, -2)$ and $(0, 3)$

Example 6  Find equation of line through $(6, -3)$ and perpendicular to $y = 5x - 3$
Definition of a Quadratic Function
A quadratic function of $x$ is a function that can be represented by an equation of the form

$$f(x) = ax^2 + bx + c$$

Where $a$, $b$, and $c$ are real numbers and $a \neq 0$

Definition of a Symmetry with Respect to a Line.
A graph is symmetric with respect to a line $L$ if for each point $P$ on the graph there is a point ‘$P$’ on the graph such that the $L$ is the perpendicular bisector of the line segment $PP'$.

Standard Form of Quadratic Equation
Every quadratic function $f$ given $f(x) = ax^2 + bx + c$ can be written in the standard form

$$f(x) = a(x - h)^2 + k$$

The graph of $f$ is a parabola with vertex $(h,k)$ and line of symmetry is $x = h$

Example 1 Given $f(x) = x^2 + 6x - 1$, complete the square to find the standard form of a quadratic function. Sketch graph and label vertex and axis of symmetry.
Finding $x$ coordinate of the vertex of a quadratic function.

Zero of a quadratic function $ax^2 + bx + c = 0$ are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2}$$

Because of symmetry of a parabola, the $x$ coordinate of the vertex is the average of the two zero’s or

$$x\text{-coord of vertex} =$$

Vertex Formula:

The function $f(x) = ax^2 + bx + c$ has vertex with coordinates $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$

Example 2 Use the vertex formula to determine the vertex of the graph of the function and write function in standard form

$$f(x) = 3x^2 - 6x + 5$$
Properties of Graphs

Symmetry

Symmetric with respect to x axis:

Symmetric with respect to y axis:

Symmetric with respect to a point or the origin:

The graph of an equation is symmetric with respect to:

- the y-axis if replacement of x with -x leaves equation unaltered. “**EVEN**” Function
- the x-axis if replacement of y with –y leaves equation unaltered
- the origin if replacement of x with –x and y with –y leaves equation unaltered. “**ODD**” Function
If $f$ is a function and $c$ is a positive constant, then:

if $c > 1$ the graph of $y = c \cdot f(x)$ is the graph of $y = f(x)$ stretched vertically by a factor of $c$ away from the x-axis.

if $0 < c < 1$ the graph of $y = c \cdot f(x)$ is the graph of $y = f(x)$ compressed vertically by a factor of $c$ toward the x-axis.

\[
\begin{align*}
    f(x) &= |x| \\
    g(x) &= 2f(x)
\end{align*}
\]

\[
\begin{align*}
    f(x) &= |x| \\
    g(x) &= \frac{1}{4}f(x)
\end{align*}
\]
If $f$ is a function and $c$ is a positive constant, then

if $c > 1$ the graph of $y = f(cx)$ is the graph of $y = f(x)$ compressed horizontally by a factor of $\frac{1}{c}$ toward $y$-axis

if $0 < c < 1$ the graph of $y = f(cx)$ is the graph of $y = f(x)$ stretched horizontally by a factor of $\frac{1}{c}$ away from the $y$-axis.

\[ f(x) = x^2 \]

\[ c = 2 \]
\[ y = f(2x) = (2x)^2 = 4x^2 \]

\[ f(x) = x^2 \]
\[ c = \frac{1}{2} \]
\[ y = f(\frac{1}{2}x) = (\frac{1}{2}x)^2 = \frac{x^2}{4} \]
Operations of Functions

Sum \((f + g)(x) = f(x) + g(x)\)

Difference \((f - g)(x) = f(x) - g(x)\)

Product \((fg)(x) = f(x) \cdot g(x)\)

Quotient \(\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad g(x) \neq 0\)

Domain of \(f + g, f - g, fg, \frac{f}{g}\):

For the functions \(f\) and \(g\), the domains of \(f + g, f - g, \text{ and } fg\)
consist of all real numbers formed by the intersection of the domain of \(f\) and \(g\). The
domain of \(\frac{f}{g}\) is the set of all real numbers formed by the intersection of domains of \(f\)
and \(g\), except for real numbers such that \(g(x) = 0\).
2.6
Examples

Let \( f(x) = x^2 - 9 \) and \( g(x) = 2x + 6 \)

Find

a) \( (f + g)(5) \)

\[
(f + g)(x) = x^2 - 9 + 2x + 6 = x^2 + 2x - 3
\]
\[
(f + g)(5) = 5^2 + 2(5) - 3 = 25 + 10 - 3 = 32
\]

b) \( (f \circ g)(-1) \)

\[
(f \circ g)(x) = (x^2 - 9)(2x + 6) = 2x^3 + 6x^2 - 18x - 54
\]
\[
(f \circ g)(-1) = 2(-1)^3 + 6(-1)^2 - 18(-1) - 54 = -32
\]

c) \( \frac{f}{g}(4) \)

\[
\frac{f}{g}(x) = \frac{x^2 - 9}{2x + 6} = \frac{(x + 3)(x - 3)}{2(x + 3)} = \frac{x - 3}{2}
\]
\[
\frac{f}{g}(4) = \frac{4 - 3}{2} = \frac{1}{2}
\]
If \( f(x) = \sqrt{x-1} \) and \( g(x) = x^2 - 4 \), find domain of \( f + g, f - g, f \cdot g, \text{ and } \frac{f}{g} \)

Domain of \( f(x) \) is

Domain of \( g(x) \) is

Domain \( f + g, f - g, f \cdot g \) is

Domain for \( \frac{f}{g} \) same as above except for values of \( x \) where \( g(x) = 0 \)

Or \( x^2 - 4 = 0 \) \( x = \pm 2 \)

(only need to consider +2)

Domain of \( \frac{f}{g} \) is
2.6
Difference Quotient

\[
\frac{f(x+h) - f(x)}{h}, h \neq 0 \text{ is called difference quotient.}
\]

**Example**
Find “difference quotient” of \( f(x) = x^2 + 7 \)

\[
\frac{f(x+h) - f(x)}{h} = \frac{[(x+h)^2 + 7] - [x^2 + 7]}{h}
\]

\[
= \frac{[x^2 + 2xh + h^2 + 7] - [x^2 + 7]}{h}
\]

\[
= \frac{x^2 + 2xh + h^2 + 7 - x^2 - 7}{h}
\]

\[
= \frac{2xh + h^2}{h} = \frac{h(2x + h)}{h} = 2x + h
\]
Definition of Composition of Two Functions
Let \( f \) and \( g \) be two functions such that \( g(x) \) is in domain of all \( f \) for all \( x \) in domain of \( g \). Then the composition of two functions, denoted by \( f \circ g \), is the function whose value at \( x \) is given by \( (f \circ g)(x) = f(g(x)) \).

#38  p. 252
\[ f(x) = 2x - 7, \quad g(x) = 3x + 2 \]
\[ (f \circ g)(x) = \]
\[ (g \circ f)(x) = \]

#40
\[ f(x) = x^2 - 11x, \quad g(x) = 2x + 3 \]
\[ (f \circ g)(x) = \]
\[ (g \circ f)(x) = \]