**10.5 Parametric Surfaces**

In 10.1 we described a space curve by vector function  \( \mathbf{r}(t) \) which had a single parameter, \( t \). We parametrically represented a curve.

Today we'll describe a parametric surface by a vector function  \( \mathbf{r}(u,v) \) which has two parameters, \( u \) and \( v \).

One parameter is needed for a curve in space and two parameters are needed to describe a surface.

So it will look like this:

\[
\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle
\]

these components are functions of 2 variables, \( u \) and \( v \)

For example: \( \mathbf{r}(u,v) = \langle \cos u, v^2 + 1, u + v \rangle, \ -1 \leq u \leq 1, \ -1 \leq v \leq 1 \)

Each choice of \( u \) and \( v \) is a point on the surface which is traced out by the tip of \( \mathbf{r} \).

Input \( (u, v) \), output \( \mathbf{r}(u,v) \).

So \( \mathbf{r}(u,v) \) is a vector-valued function defined on a region (the domain) in the \( uv \)-plane.

Before, the domain of \( \mathbf{r}(t) \) was the \( t \) values.
Now we’re going to find parametric representations for surfaces.

Let’s start with a plane:

We’ve already defined a plane as $ax + by + cz + d = 0$ or $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ using normal vector $\mathbf{n} = \langle a, b, c \rangle$.

If we want to find a vector function that represents the plane that passes through point $(x_0, y_0, z_0)$ with position vector $\mathbf{r}_0$ and that contain two non parallel vectors $\mathbf{a}$ and $\mathbf{b}$.

I can define any point on the plane by a scalar multiple

$$u\mathbf{a} + v\mathbf{b}$$

In 3-dimensions we need a starting point $(x_0, y_0, z_0)$ from position vector $\mathbf{r}_0$ and then we add on the points we defined from above. This describes every point on the surface (plane).

A parametric representation of a plane in 3-dimensions is

$$\mathbf{r}(u, v) = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b}$$

$$\mathbf{r}(u, v) = \langle x_0, y_0, z_0 \rangle + u\langle a_1, a_2, a_3 \rangle + v\langle b_1, b_2, b_3 \rangle$$

$$\mathbf{r}(u, v) = \langle x_0 + ua_1 + vb_1, \ y_0 + ua_2 + vb_2, \ z_0 + ua_3 + vb_3 \rangle$$

** These parametric equations of surfaces will become very important later on!

<table>
<thead>
<tr>
<th>Parametric Equation of a Plane</th>
</tr>
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<tbody>
<tr>
<td>$x = x_0 + ua_1 + vb_1$</td>
</tr>
<tr>
<td>$y = y_0 + ua_2 + vb_2$</td>
</tr>
<tr>
<td>$z = z_0 + ua_3 + vb_3$</td>
</tr>
</tbody>
</table>

Example: Can you identify the surface with the given vector equation $\mathbf{r}(u, v) = \langle u + v, \ 3 - v, \ 1 + 4u + 5v \rangle$?

$$\mathbf{r}(u, v) = \langle u + v, \ 3 - v, \ 1 + 4u + 5v \rangle$$

$$= \langle 0 + 1u + 1v, \ 3 + 0u - 1v, \ 1 + 4u + 5v \rangle$$

$$= \langle 0, \ 3, \ 1 \rangle + \langle 1u + 1v, \ 0u - 1v, \ 4u + 5v \rangle$$

$$= \langle 0, \ 3, \ 1 \rangle + u\langle 1, \ 0, \ 4 \rangle + v\langle 1, \ -1, \ 5 \rangle$$
To use more conventional formulas of equations of planes such as \( ax + by + cz + d = 0 \) or \( a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \) where the normal vector is \( \vec{n} = \langle a, b, c \rangle \) you would take the cross of vectors \( \vec{a} \) and \( \vec{b} \).

\[
\vec{a} \times \vec{b} = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
1 & 0 & 4 \\
1 & -1 & 5
\end{vmatrix} = (4, -1, -1)
\]

so \( 4(x - 0) - (y - 3) - (z - 1) = 0 \) or \( 4x - y - z = -4 \)

There is an easier way to parametrize a plane other than \( \vec{r}(u, v) = \langle x_0 + ua_1 + vb_1, y_0 + ua_2 + vb_2, z_0 + ua_3 + vb_3 \rangle \).

In general, a surface given as the graph of a function of \( x \) and \( y \), in the form \( z = f(x, y) \), can always be regarded as a parametric surface by taking \( x \) and \( y \) as parameters and writing \( x = x, y = y, z = f(x, y) \).

Solve \( 4x - y - z = -4 \) for \( z \) and you get \( z = 4x - y + 4 \) so let \( x = x, y = y, z = 4x - y + 4 \) or \( \vec{r}(x, y) = \langle x, y, 4x - y + 4 \rangle \)

** This will be very useful later!

What if we solved for \( y \) in terms of \( x \) and \( z \) instead?

What if we solved for \( x \) in terms of \( y \) and \( z \) instead!

Example 2:
Find a parametric representation for the elliptic paraboloid, \( z = x^2 + y^2 \).

Just leave it with parameters \( x \) and \( y \), there's no need to change it to \( u \)'s and \( v \)'s.

Parametric representations are also called parametrizations. They are not unique; there are numerous ways to parametrize surfaces.

Example 3:
Find a parametric representation for the surface \( z = 2\sqrt{x^2 + y^2} \). This is the top half of cone \( z^2 = 4x^2 + 4y^2 \).
To parametrically represent the part of cone $z = 2\sqrt{x^2 + y^2}$, you probably did Case 1:

**Case 1** – You chose $x$ and $y$ as parameters by letting $x = x, y = y, z = 2\sqrt{x^2 + y^2}$

so the vector equation is $\mathbf{r}(x,y) = \langle x, y, 2\sqrt{x^2 + y^2} \rangle$, a parametric representation of the cone.

You could do this instead:

**Case 2** – Choose $r$ and $\theta$ as parameters (polar coordinates). A point $(x, y, z)$ on the cone satisfies

$x = r \cos \theta, \quad y = r \sin \theta, \quad \text{and} \quad z = 2\sqrt{x^2 + y^2} = 2\sqrt{r^2} = 2r$

so the vector equation is $\mathbf{r}(r, \theta) = \langle r \cos \theta, r \sin \theta, 2r \rangle$ where $r \geq 0, \quad 0 \leq \theta \leq 2\pi$.

HW #20:
Find a **parametric representation** for the lower half of the ellipsoid $2x^2 + 4y^2 + z^2 = 1$.

*You don’t need to follow the solution’s manual on* HW #’s 3-6 where you are asked to **Identify the surface with the given vector equation**. You could just do as we did before and just eliminate the parameter, except now we need to eliminate both parameters $u$ and $v$ instead of just $t$.

We have much more practice seeing the surface in terms of $x, y$ and $z$ because we have a lot more experience with them. Set the stuff in the $x$-component spot equal to $x$, set the stuff in the $y$-component spot equal to $y$ and set the junk in the $z$-component equal to $z$ and then work your magic trying substitution, elimination, or using some identities like $\sin^2 t + \cos^2 t = 1$ to get rid of parameters $u$ and $v$ and create $x$'s, $y$'s and $z$'s. Just eliminate both parameters $u$ and $v$ and put it in terms of $x, y$ and $z$ so we’ll have a chance at recognizing it.

HW #4.
**Identify the surface** of the vector equation $\mathbf{r}(u,v) = \langle 2 \sin u, 3 \cos u, v \rangle, \quad 0 \leq v \leq 2$. (notice the restriction placed on parameter $v$)
Show them this surface on *Mathematica* I have on our website.

**Grid Curves on a Parametric Surface** (sometimes called grid lines) – found by holding parameters $u$ and $v$ constant one at a time.

Example 1. **Find the grid curves** associate with $u$ being held constant $\mathbf{r}(u,v) = \langle 2\sin u, 3\cos u, v \rangle$.

Find the grid curves associate with $v$ being held constant $\mathbf{r}(u,v) = \langle 2\sin u, 3\cos u, v \rangle$.

**TEC 10.5** has crazy surfaces with crazy grid curves to see.

***Mathematica Lab 10.5 –***

I want you to do 8, 9, and 10 on *Mathematica*.

We will do #7 right now so you’ll know what I’m looking for:

- Use *Mathematica* to graph the parametric surface. Print out a nice picture.
- Copy and Paste the surface to manipulate it into a desired position so you can see the various grid curves.
- Get a printout and indicate which grid curves have $u$ constant and indicate which grid curves have $v$ constant by drawing on the printout with two different colored pens or markers. Give me a key (Example: the red pen are the grid curves associated with $u$ being constant and the green pen are the grid curves associated with $v$ being constant).
- Be sure to label your $x$, $y$, and $z$-axis on each graph.
- Be very clear.
General Example 4:
Find a parametric representation of a sphere \( x^2 + y^2 + z^2 = a^2 \).

remember spherical coordinates \((\rho, \theta, \phi)\)
\[
\begin{align*}
x &= \rho \sin \phi \cos \theta, \\
y &= \rho \sin \phi \sin \theta, \\
z &= \rho \cos \phi
\end{align*}
\]
Hopefully you came up with \( \tilde{r}(\theta, \phi) = (a \sin \phi \cos \theta, \ a \sin \phi \sin \theta, \ a \cos \phi) \), \( 0 \leq \theta \leq 2\pi, \ 0 \leq \phi \leq \pi \).

Spherical coordinates, \((\rho, \theta, \phi)\), where spheres are represented \( \rho = a \).

We **choose** \( \theta \) and \( \phi \) as parameters in spherical coordinates, substitute \( \rho = a \) into the conversion equations
\[
x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi,
\]
then
\[
\tilde{r}(\theta, \phi) = (a \sin \phi \cos \theta, \ a \sin \phi \sin \theta, \ a \cos \phi), \quad 0 \leq \theta \leq 2\pi, \ 0 \leq \phi \leq \pi.
\]

So the parameter domain is the rectangle \( D = [0, 2\pi] \times [0, \pi] \)

**Let's look at the grid curves:**

If we **hold** \( \theta \) **constant**, \( \theta = c \), look to where there is our variable \( \phi \).

\[
\tilde{r}(c, \phi) = (a \sin \phi \cos c, \ a \sin \phi \sin c, \ a \cos \phi)
\]

There's \( \phi \) in each component.

The \( x, y, \) and \( z \) components all have to with \( \phi \) which comes off the \( z \)-axis. So these grid curves are effected by \( x, y, \) and \( z \) all at the same time. They are the meridians which connect the north and south poles.

If we **hold** \( \phi \) **constant**, \( \phi = c \), then look to where parameter \( \theta \) is located.

\[
\tilde{r}(\theta, c) = (a \sin c \cos \theta, \ a \sin c \sin \theta, \ a \cos c)
\]

\( \theta \) is **only located in the \( x \) \& \( y \) components**

So our **grid curves will be parallel to \( xy \)-plane only.**

\[
x = a \sin c \cos \theta \quad \left( \frac{x}{a \sin c} \right)^2 + \left( \frac{y}{a \sin c} \right)^2 = 1
\]
\[
x^2 + y^2 = (a \sin c)^2
\]

Grid curves are circles parallel to \( xy \)-plane (horizontal). Circles of constant latitude (like the equator).

If we **hold** \( \theta \) **constant**, \( \theta = c \), then look to where parameter \( \phi \) is located.

\[
\tilde{r}(c, \phi) = (a \sin \phi \cos c, \ a \sin \phi \sin c, \ a \cos \phi)
\]

There's \( \phi \) in each component. The \( x, y, \) and \( z \) components all have \( \phi \). So these grid curves are effected by \( x, y, \) and \( z \) all at the same time. They are the meridians which connect the north and south poles (remember \( \phi \) comes off the \( z \)-axis).

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Often we only need to look at a piece of a surface instead of the whole thing. It's easy to put restrictions on parameters. For example, how could we restrict a sphere to the 1st octant? \( \theta \leq \leq \), \( \phi \leq \leq \).

** This will be important later!

We will not cover Surfaces of Revolution.