12.2 Iterated Integrals

Super fun section to do, if you’re good at integrating!

Be able to integrate 1) Mentally, 2) using U-Substitution, and 3) using By-Parts!

\[
\int_{0}^{1} \int_{0}^{1} 3x^2 y \, dy \, dx
\]

Called an iterated integral. Here we integrate with respect to \( y \) first.

\[
\int_{0}^{1} \left[ \int_{0}^{1} 3x^2 y \, dy \right] \, dx
\]

We work from the inside out: \( y \)'s \([1,3]\), then later our \( x \)'s \([0,2]\). So our rectangular region

\[
R = [0,2] \times [1,3]
\]

Our function is \( f(x,y) \) or \( z = 3x^2 y \).

We’re finding the exact volume of the solid that’s below surface \( z \) and above region \( R \) (assuming \( z \geq 0 \), this represents total volume).

Riemann Sums were an approximation, we added up a certain number of columns to approximate the volume. Now we let the number of columns approach infinity and calculate the exact volume by calculating the double integral (iterated integral).

Example 1:

\[
\int_{0}^{1} \int_{0}^{1} 3x^2 y \, dy \, dx =
\]
Now try Example 2:
\[ \int_{1}^{2} \int_{0}^{3} 3x^2 y \, dx \, dy \]
We treat \( x \) as an independent variable and \( y \) as a constant.

**TEC 12.2** shows a nice illustration of what’s going on here.

\[
\int \int \left( \frac{f(x,y)}{\text{area of a sheet}} \right) \, dx \quad \text{set } x \text{ constant, } x = 0 \text{ (1 sheet), } x = 1 \text{ (1 sheet)} \ldots \text{we get the total volume by summing up all the areas of each sheet. The purple sheets are } A(x). \\
OR:\n\int \int \left( \frac{f(x,y)}{\text{area of a sheet}} \right) \, dy \quad \text{set } y \text{ constant, } y = 0 \text{ (1 sheet), } y = 1 \text{ (1 sheet)} \ldots \text{we get the total volume by adding up the areas of each sheet. The blue ones.}
\]

*Here it doesn’t matter which order we integrate, 1\(^{st}\) with respect to \( x \), then \( y \), or vice versa.*

**Guido Fubini** gets this theorem named after him.

*This only happens when the limits of integration are constants*

The limits of integration must be numbers (not variables) in order to be able to integrate both ways.

Next section:
\[ \int_{0}^{2} \int_{x}^{x^2} \]

What will your answer look like if you integrate in this order \[ \int_{0}^{2} \int_{x}^{x^2} \] ? What about here \[ \int_{x}^{2} \int_{0}^{x} \] ?

It might be easier one way or the other on certain problems, try to think ahead and choose the easiest path. \( dy \, dx \) where you start with \( y \) being your variable and \( x \) as a constant or \( dx \, dy \). It all depends on what you are integrating.

**You need to integrate all HW manually**, but be able to integrate using your graphing calculator also!

On my TI-89 I would enter:
\[ \int \left( \int (3 \times x \land 2 \times y, y, 1, 3), x, 0, 2 \right) = 32 \]
Example 3 (No calculators at first!):
Evaluate \[\iint_{R} y \sin(xy) \, dA\], where \(R = [1,2] \times [0,\pi]\).

What could this look like?
How can volume = 0?
This problem was example 3 on Page 840 or 841, see Figure 4.
Volume above \(R\) = volume below \(R\), so they cancelled each other out. Also, look at their solution 2; integrated with respect to \(y\) first, wow, way harder! Integrate by parts twice.
This is what I mean by look ahead and choose the easiest way to integrate; first with respect to \(x\) and then \(y\) OR first with respect to \(y\) and then \(x\). One way might be WAY easier! You can choose either way only when the limits of integration are all constants.

**Another helpful trick to use in the special case where you can write your function \(z\) in a product of (function of \(x\)) \cdot (function of \(y\)) you can separate them. Here’s an example and why it works:

(Past Example)
\[
\begin{align*}
\frac{3}{2} \int_{1}^{2} \frac{3}{2} \int_{0}^{3} x^2 y \, dx \, dy &= \frac{3}{2} \left[ \int_{1}^{2} 3x^2 y \, dx \right] \, dy \\
&= \int_{0}^{2} \left[ \int_{0}^{3} x^2 dx \right] \, dy \\
&= \int_{0}^{2} 3x^2 \, dy \\
&= x^3 \bigg|_{1}^{2} \\
&= 8 - 0 \\
&= 8 \\
&= \left( \frac{9}{2} - \frac{1}{2} \right) \\
&= \frac{32}{4} \\
&= 32
\end{align*}
\]
*This also only works if all 4 limits of integration are constants (next section they may not be), but be on the lookout for this, it makes integrating pretty easy!
Don’t forget the order to do $\iint$, first try it mentally, then I usually look at using $u$-substitution, and then using integration by parts. I will be testing your integration abilities. We won’t be using the charts with all of the formulas too much.

Example 4:

$$\iint_{R} \frac{x}{1+xy} \, dA,$$

$R = [0,1] \times [0,1]$
Example 5 (just set it up):
Find the volume of the solid in the first octant bounded by \( z = 9 - y^2 \) and plane \( x = 2 \).

I want you to sketch the region we’re integrating over (a very important concept as we move on).

Example 6:
Sketch the solid whose volume is given by the iterated integral, \( \int_0^1 \int_0^1 (2 - x^2 - y^2) \, dy \, dx \).