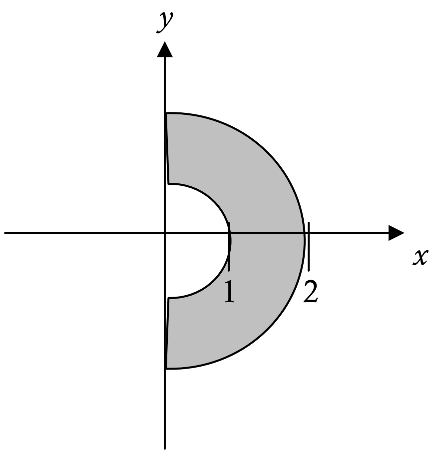
12.4 **Double Integrals in Polar Coordinates**

Do Integration manually on HW!

When describing <u>regions,</u> in polar coordinates is way easier than rectangular, we will always use them in calculating double integrals.
Let's say the shaded region is the region that you are integrating over.



$x^2 + y^2 = 1^2$ $\qquad$ $x^2 + y^2 = 2^2$

How could you use Type I to represent this region ? $\uparrow$

How could you use Type II to represent this region ? $\rightarrow$

But, it's easy in Polar

$\underbrace{x^2 + y^2}_{r^2} = 1 \qquad \underbrace{x^2 + y^2}_{r^2} = 4$

$r = 1 \qquad r = 2$

r is always positive in polar

Polar arrows shoot out from the origin.
Every polar arrow enters $r = 1$ and exits $r = 2$.

$R = \left\{ (r,\theta) \mid \ 1 \le r \le 2, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \right\}$

This is just our region in the *xy*-plane, we still need to project it up to the surface so our double integral would look something like:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_1^2 (\,function\ f\,)\ r\ dr\,d\theta$$

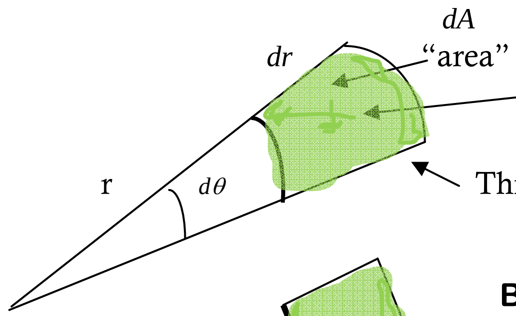
I'll explain this extra r.

Last time in rectangular we looked at columns and added them all up.
This time in polar we look at polar rectangles (wedges.)

Page 853, figures 3 and 4.

We'll take our entire region, R, and divide it up into polar subrectangles and add them up with a Riemann Sum. As we let the number of subrectangles $\to \infty$, it becomes an exact instead of approximate.

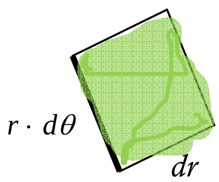
$dx \quad dy$ (before)



length·width

$r\ d\theta \cdot (dr) = r\,dr\,d\theta$

Think of this as a (polar) rectangle

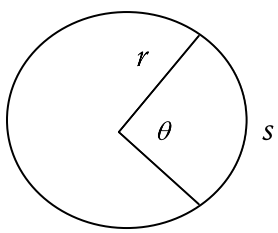


Before, $dA = dx\,dy$ or $dy\,dx$ **that's how we calculated the area of the rectangles.**

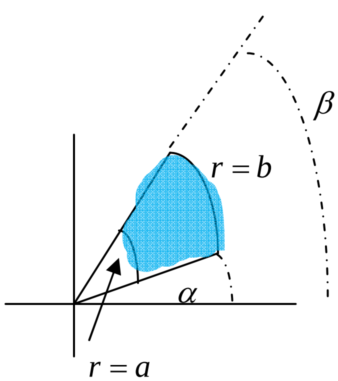
Now **

$$dA = r\,dr\,d\theta$$

$\frac{s}{2\pi r} = \frac{\theta}{2\pi}$ so $s = r\theta$ (in radians)



$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\}$ We must change our fn. to polar.



$$V = \iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\ r\,dr\,d\theta$$

"alpha"
"beta"

surface still

new polar attachment $r\,dr\,d\theta$

All polar arrows start from the origin and enter $r = a$ and exit $r = b$ and are contained from $\alpha \le \theta \le \beta$.

So our problem

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_1^2 \underbrace{f(r\cos\theta, r\sin\theta)}\ r\,dr\,d\theta$$

The functions will need to be converted from rectangular to polar so convert all *x*'s and *y*'s to *r*'s and θ's by using:

$\left.\begin{array}{l} x = \cos\theta \\ y = r\sin\theta \end{array}\right\}$ along with $x^2 + y^2 = r^2$ and $\tan\theta = \frac{y}{x}$.

If, for example, our $f(x,y)=3x+y^2$. The question might read:

Evaluate $\iint_R \left(3x+y^2\right)dy\,dx$ using polar coordinates. You combine this extra *r* with your function before you integrate.

$$\iint_R \left(3x+y^2\right)dy\,dx$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_1^2 \left(3r\cos\theta+r^2\sin^2\theta\right)\, r\,dr\,d\theta$$

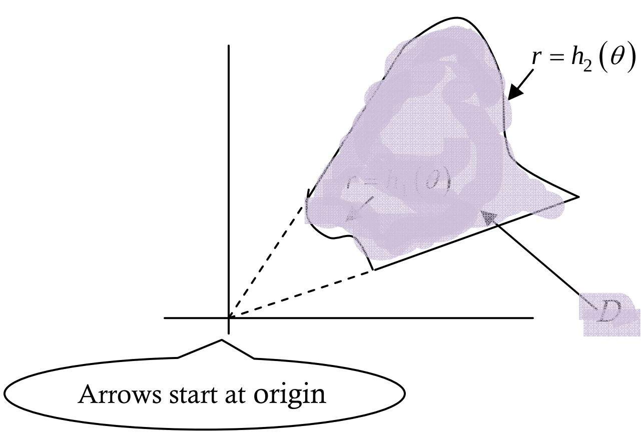
$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_1^2 \left(3r^2\cos\theta+r^3\sin^2\theta\right)\, dr\,d\theta$$

$$\approx\ 22$$

(on the graphing calculator)
Don't have time to solve this one. It's a pain to solve manually (**the HW problems can and should be done manually**).

Be able to integrate with your calculator also, so check a couple of them as you are doing them manually; especially the even-numbered problems.

The equations for *r* don't have to be just constants, they may have variable θ in them so you may have all polar arrows start from the origin and enter $r=h_1(\theta)$ and exit $r=h_2(\theta)$ contained by $\alpha\le\theta\le\beta$.



If $D=\{(r,\theta)\mid\ \alpha\le\theta\le\beta,\ \ h_1(\theta)\le r\le h_2(\theta)\}$, then

$$\int\int f(x,y)\qquad dA$$

$$\int_\alpha^\beta\int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta,r\sin\theta)\ r\,dr\,d\theta$$

HW #20:

Use polar coordinates to find the volume of the solid bounded by the paraboloid $z = 1 + 2x^2 + 2y^2$ and the plane $z = 7$ in the first octant.

These are difficult to draw, but a good drawing really helps to eliminate mistakes.
The paraboloid is shifted up 1 unit. Floating above the *xy*-plane.

Where does the plane intersect the paraboloid?

This was also in the last section. I didn't mention it in 12.3

You could use a double integral to find the area (not the volume) of a region, *D*, **IF** you let the function, $z = 1$.

$$\text{Box} = l \cdot w \cdot h \quad \leftarrow \text{volume}$$
$$= l \cdot w \cdot 1$$
$$= l \cdot w$$
$$= \text{ an area}$$

$f(x, y)$ or z ↓

$$A(D) = \iint_D 1 \quad dA$$

When we integrate the constant function $f(x, y) = 1$ over a region *D*, we get the area of *D*.
You're taking the third dimension away so volume turns into area. Your height is 1, so you are multiplying the area of region D times 1, so you get the area of region D.
This idea will come up again!

Example 1:
Find the area inside spiral $\theta = r$, where $0 \le \theta \le \pi$.

<u>The worksheet online is due in a week</u>:
I ask you to compute an area numerous ways:

1) **Using geometry,**
2) **Using Single variable (one integral from your old Calculus II ideas),**
3) **Using double integrals (both** $\iint dxdy$ and $\iint dydx$, and also using $\iint r\, dr\, d\theta \ldots$ **fun!)**
4) **Using polar coordinates (not very fun on this one)**

It's nice to see the same question approached differently.

Make sure all of your answers match each other.

Example 2:

Use polar coordinates to find the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$.

A little hard to visualize/draw the 3-dimensional figure, but very necessary!
Necessary to draw the region over integration.

If done manually, you could split it up into the product of two integrals because the limits of integration are all constants. Your choice.

Let's practice one more converting to polar.

Example 3:

Evaluate the iterated integral, $\int_0^1 \int_y^{\sqrt{2-y^2}} (x+y)\, dx\, dy$, by converting to polar .

If you need to integrate $\cos^2 x$ or $\sin^2 x$ I'll expect you to be able to integrate them manually. Remember to use half-angle formulas and substitute

$$\cos^2 x = \frac{1+\cos 2x}{2} \quad \text{or} \quad \frac{1}{2}+\frac{\cos 2x}{2}.$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \quad \text{or} \quad \frac{1}{2}-\frac{\cos 2x}{2}$$

Once you make this substitution, the integration is easy.