Chapter 13 – Vector Calculus

We will be studying the calculus of vector fields. Vector fields are functions that assign vectors to points in space.

We will define line integrals, which can be used to find work done by a force field in moving an object along a curve.

Once we learn about line integrals, we'll be able to study Green's Theorem, Stoke's Theorem, and lastly the Divergence Theorem.

After line integrals we will look at surface integrals, which can be used to find the rate of fluid flow across a surface.

First, we start with vector fields in section 13.1.

Along the way, if you have experience in any applications that go along with the mathematics we are studying, feel free to share!

13.1 Vector Fields

(we'll start with 2-dimensional vector fields)

Examples of <u>vector</u> <u>fields</u>:

<u>Figure 1a&b</u> – Air velocity vectors that indicate wind <u>speed</u> <u>and</u> <u>direction</u> . Can you tell the where the wind speeds were the fastest? **Thicker** arrows indicate faster wind speeds (we will be drawing vectors with a longer length instead of thickness). They are also using different colors to indicate different speeds. This is a velocity vector field.

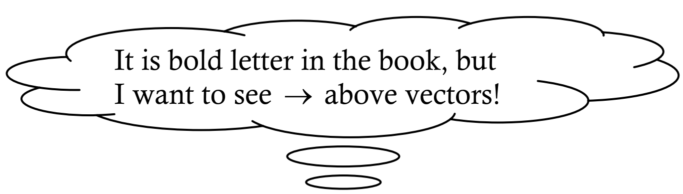
<u>Figure 2</u> – Ocean currents (a)
Air flow past an inclined airfoil (b) ← Both velocity vector fields.

<u>Figure 13</u> – Velocity field in fluid flow – where is the speed of the fluid the fastest?
(the longer vectors are inside the smaller radius which indicates their speed is faster)

<u>Figure 14</u> – Gravitational force field –associates a force vector with each point in space. Newton's Law of Gravitation.

<u>Example</u> – Electric force fields – electric charge Coulomb's Law

<u>Figure 15</u> – our old gradient vector, $\nabla f = \langle f_x, f_y \rangle$, "dell f" is a gradient vector field. Figure 15 shows a **contour map of the function in red** with the **gradient vector field in blue**. Notice again the gradient vectors are perpendicular to the level curves. Notice the gradient vectors long where the level curves are close together (short where the level curves are far apart). Closely spaced level curves indicate a steep graph and the length of the gradient vector is a value of the directional derivative of the function, so those values will be larger; long gradient vectors. Level curves far apart indicate a less steep slope so the value of the directional derivative will be smaller; short gradient vectors.



<u>Definition</u>: A vector field in 2-dimensions is a function $\vec{F}$ that assigns to each point (x, y) in the domain a 2-dimensional vector $\vec{F}(x, y)$.

*the Domain is a set of <u>points,</u> (x, y), the Range is a set of <u>vectors</u>, $\vec{F}(x, y)$.

Written in terms of its component functions, P and Q, $\vec{F}(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j}$ or $\langle P(x, y), Q(x, y)\rangle$.
P and Q are scalar fields, not vector fields.

The best way to picture a vector field is to draw the arrow representing the vector, $\vec{F}(x,y)$, starting at the point (x,y). You obviously can't do it for all points (x,y), but you get a pretty good idea what the vector fields look like by doing some points.

HW#4:
Sketch the vector field $\vec{F}(x,y)=\langle x-y,\ x\rangle$.

Notice the different length of vectors?
What would the length be?
How could we represent it?

Why are the vectors along the line $y=x$ all vertical?

These are a pain to do manually. Technology will come in handy here. But that's all these vector fields are; that's all they do; start at the given point and draw the corresponding vector.

Figure 6: $\vec{F}(x,y)=\langle -y,x\rangle$ This could be representing the counter-clockwise rotation of a wheel.
Quadrant I

Quadrant II

Quadrant III

Quadrant IV

Why do the vectors get longer as you move away from the origin?

What's going on at the origin?

What's happening along *x*-axis and why?

What's happening along *y*-axis and why?

<u>3-Dimensional</u> <u>Vector</u> <u>Fields</u> – hard to do by hand, you get to do 1 for HW
Technology will come in handy here…
Figures 10, 11, and 12 – you get a better feel for them if you can rotate them around.
What's the difference in formulas of Figures 10 and 11?
What's the difference in the look of the vector fields?

$\vec{F}(x,y,z)=\langle y,-2,\ x\rangle$

y component is fixed (-2)

Compare the two vector fields. Can you see in Figure 11 that all of the vectors point in the general direction of the negative *y*-axis (because of the -2)?

Maybe you can see it better on **TEC 13.1 3D Vector Fields, Field 4.**
As I rotate to the *xy* or *zy*-plane you can see all vectors points in the general direction of the negative negative *y*-axis.

Gradient Fields:

Our gradient f, ∇f, "dell f" from 11.6 are vector fields.

$\nabla f(x,y)=\langle f_x, f_y\rangle$ partial derivatives

$\nabla f=(x,y,z)=\langle f_x, f_y, f_z\rangle$

Gives the gradient vector at each point (x,y).

Remember Figure 15 at the beginning of the lesson showing ∇f is perpendicular to the two-dimensional level curves? **Gradient vectors are perpendicular to the level curves.** ∇f is also perpendicular to the three-dimensional level surfaces (perpendicular to the tangent plane at the point of tangency). If they'd like to see more visuals, show them p.795 Fig.7, p.796 Fig.9, p.797 Fig. 10, and p.798 Fig. 11.

Reminder: gradient vectors are long where the level curves are close together (short where the level curves are farther apart). Closely spaced level curves indicate a steep graph (steep graph means a greater slope (value), so a larger vector). The length of a gradient vector is the value of the directional derivative of *f*.

In order to sketch the gradient vector field ∇f of f, first find $\nabla f(x,y)=\langle f_x, f_y\rangle$ and then sketch it the same way we did HW #4.

Let's use *Mathematica* on this next one so you can do #27 on HW.
Example 1:
Find the gradient vector field ∇f of $f(x,y)=\frac{1}{4}(x+y)^2$.

We are going to plot the gradient vector field along with the contour plots.

$f[x,y]:\frac{1}{4}*(x,y)\wedge 2$
Contour Plot$\left[f[x,y],\ \{x,-3,3\},(y,-3,3)\text{ contour, shading}\rightarrow\text{False}\right]$

If you're standing on a contour line and walk perpendicular $(\perp)$ to it, you'll increase $(\uparrow)$ or decrease $(\downarrow)$ the greatest.

Needs ["Vector Field Plots"]
⟨⟨ Vector Field Plots
Gradient Field Plot $\left[f[x,y],\{x,-3,3\},\{y,-3,3\}\right]$
Show [%, % ,%]

You'll do HW #27 using *Mathematica*. Just bring up my example and adjust.

***Notice the gradient vectors are orthogonal to the contour lines!

A few Terms:

A vector field is called conservative if it's a gradient of some scalar function. In other words, if there exists a function f such that $\vec{F} = \nabla f$.

In this case, function f is called a potential function of $\vec{F}$.

Not all vector fields are conservative. I guess it's common in physics (gravitational fields).

We'll find out later on in the chapter how to tell whether or not a vector field is conservative.

Physics students can enjoy reading examples 3, 4, & 5 pertaining to fluid flow, Newton's Law of Gravitation and Electric Charge.