CIRCUITS II

Chapter 12: Introduction to the Laplace Transform

Definition of the Laplace Transform

$$\boxed{\underbrace{\mathcal{L}\{f(t)\}}_{\text{The Laplace Transform of } f(t)} = \int_0^\infty f(t)e^{-st}\,dt}$$

Alternate terminology

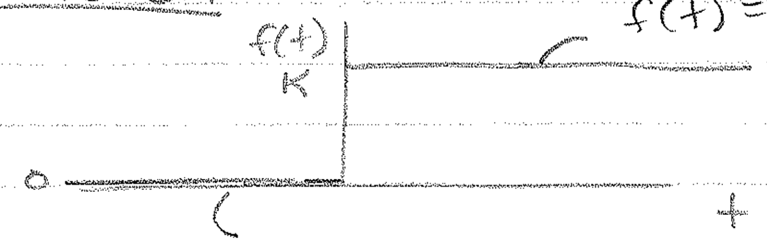
$$F(s) = \mathcal{L}\{f(t)\}$$

Functional Transform: Laplace transform of a function

Operational Transform: Mathematical Properties of the Laplace Transform (i.e. the derivative)

Step Function

$f(t) = K$ for $t > 0$



$f(t) = 0$ for $t < 0$

Step Function

$$K\,u(t)$$

Definition

$$K\,u(t) = 0 \quad \text{for } t < 0$$

$$K\,u(t) = K \quad \text{for } t > 0$$

$K\,u(t)$ is undefined at $t = 0$

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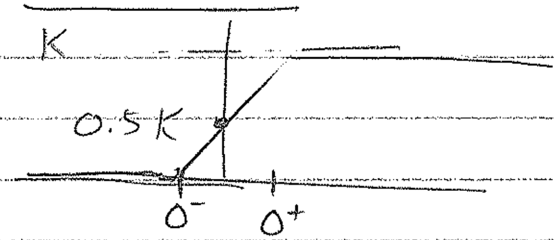
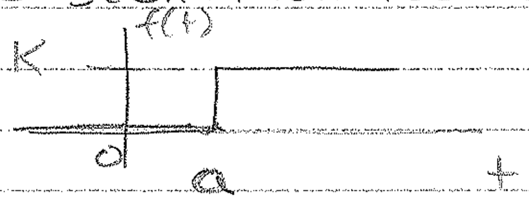
If it is necessary to have $Ku(t)$ defined at $t=0$, assume linear relation



$$K\,u(0) = 0.5K$$

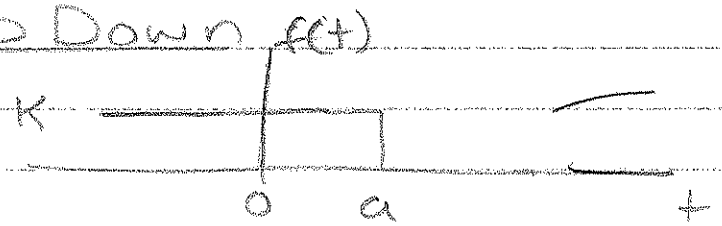
Discontinuities at times other than zero



$Ku(t-a)$

$Ku(t-a) = 0$ for $t<a$
$Ku(t-a) = K$ for $t>a$

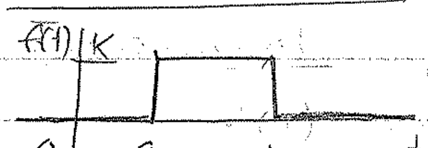
Step Down



$Ku(a-t)$

$Ku(a-t) = 0$ for $t>a$
$Ku(a-t) = 0$ for $t<a$

Finite Pulse



$K[u(t-a) - u(t-b)]$

$K[u(t-a)-u(t-b)] = 0 \quad t<a$
$K[u(t-a)-u(t-b)] = K \quad a<t<b$
$K[u(t-a)-u(t-b)] = 0 \quad b<t$

Use the step function to turn on/off the function

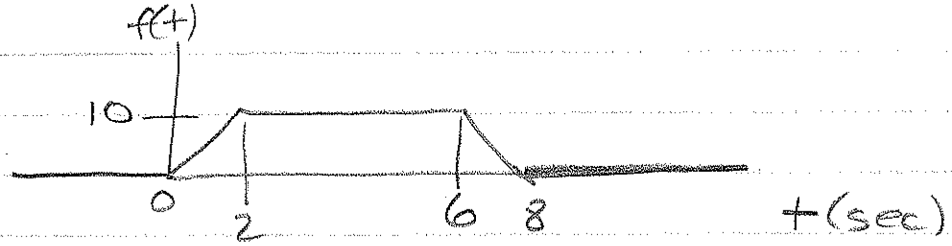
Drill Exercise 12.1

1

Drill Exercise 12.1

Use Step Functions to write the expression for each function

(a)



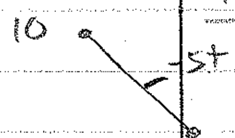
For $0 < t < 2$ sec

$$f(t) = \frac{10}{2}t = 5t \qquad f_1(t) = 5t\left[u(t) - u(t-2)\right]$$

For $2 < t < 6$ sec

$$f(t) = 10 \qquad f_2(t) = 10\left[u(t-2) - u(t-6)\right]$$

For $6 < t < 8$ sec



$$f(t) = 40 - 5t \qquad f_3(t) = (40-5t)\left[u(t-6) - u(t-8)\right]$$

$f(t) = mt + b$

$f(t) = -5t + b$

at $t = 6$ sec

$f(t) = 10$

$10 = -30 + b$

$b = 40$

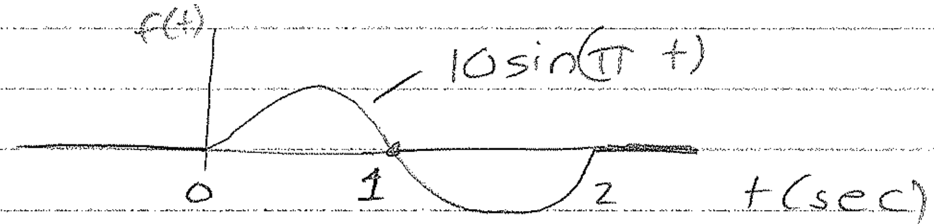
$$f(t) = f_1(t) + f_2(t) + f_3(t)$$

$$\boxed{f(t) = 5t\left[u(t) - u(t-2)\right] + 10\left[u(t-2) - u(t-6)\right] + (40-5t)\left[u(t-6) - u(t-8)\right]}$$

(2)

Drill Exercise 12.1

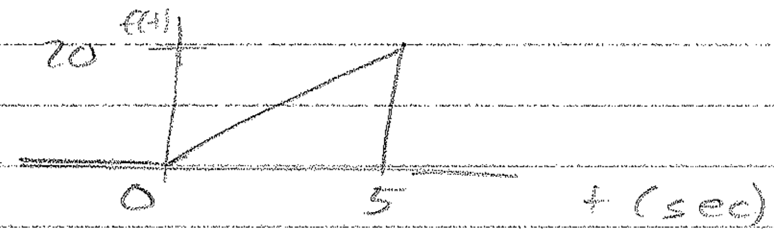
(b)



For $0 < t < 2$ $\quad f(t) = 10\sin(\pi t)$

$$\boxed{f(t) = 10\sin(\pi t)\left[u(t) - u(t-2)\right]}$$

(c)



for $0 < t < 5$ sec

$f(t) = 4t$

$$\boxed{f(t) = 4t\left[u(t) - u(t-5)\right]}$$

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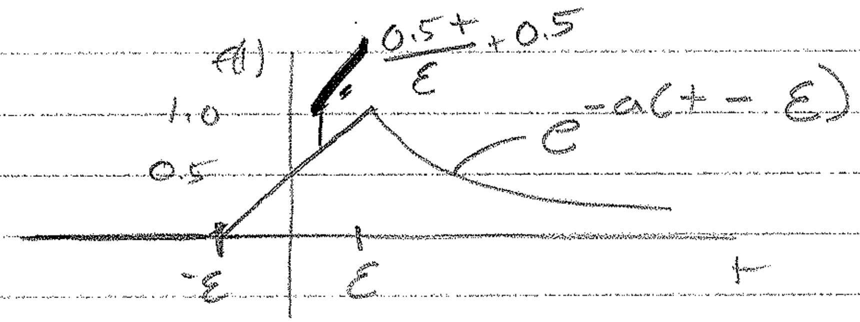
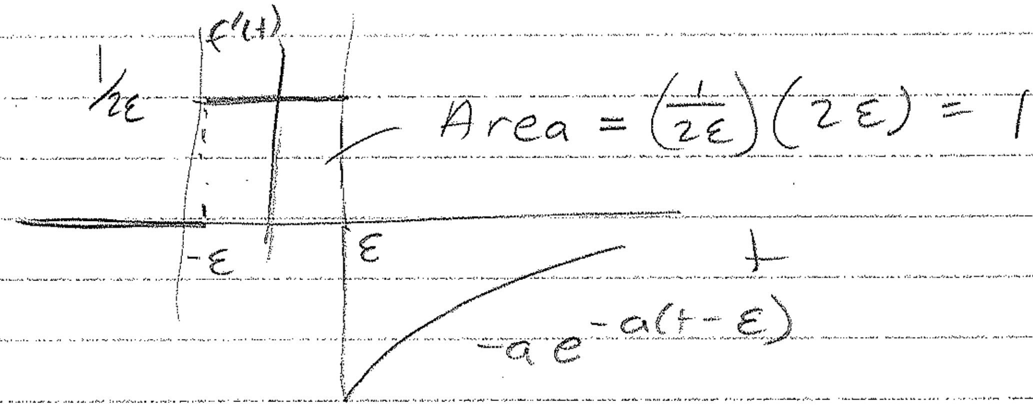
Impulse Function

Impulse Signal - Infinite amplitude
- zero duration





As $\varepsilon \to 0 \Rightarrow f'(t)$ between $-\varepsilon$ & $+\varepsilon$ approaches the unit impulse function

$$f'(0) \longrightarrow \underbrace{\delta(t)}_{\text{unit impulse function}} \text{ as } \varepsilon \to 0$$

If area is a different value than 1

impulse function $\Rightarrow \underbrace{K}_{\text{area}}\delta(t)$

Mathematically,

$$\boxed{\begin{gathered}\int_{-\infty}^{\infty} K\,\delta(t)\,dt = K \\ \delta(t) = 0 \text{ for } t \neq 0\end{gathered}}$$

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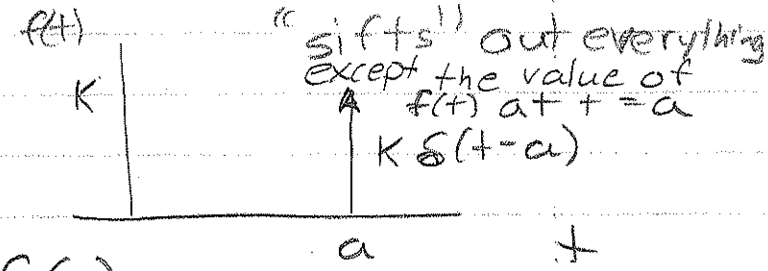
Sifting Property of the impulse function



$$\int_{-\infty}^{\infty} f(t)\,\delta(t-a)\,dt = f(a)$$

f(t) is continuous at t = a

$$\delta(t-a) = 0 \quad \text{for} \quad t \neq a$$

$$I = \int_{-\infty}^{\infty} f(t)\,\delta(t-a)\,dt = \int_{a-\varepsilon}^{a+\varepsilon} f(t)\,\delta(t-a)\,dt$$

$$= \int_{a-\varepsilon}^{a+\varepsilon} f(a)\,\delta(t-a)\,dt = f(a)\int_{a-\varepsilon}^{a+\varepsilon} \delta(t-a)\,dt$$

$$= f(a)$$

Definition of the Laplace Transform

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)\,e^{-st}\,dt$$

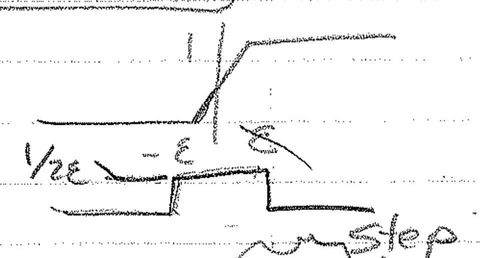
$$\mathcal{L}\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t)\,e^{-st} = \int_{0^-}^{\infty} \delta(t)\,e^{-s(0)}\,dt$$

$$= \int_{0^-}^{\infty} \delta(t)\,dt \Rightarrow \boxed{\mathcal{L}\{\delta(t)\} = 1}$$

ASA

$$\boxed{\mathcal{L}\{\delta'(t)\} = s}$$

$$\boxed{\mathcal{L}\{\delta^n(t)\} = s^n}$$



$$\delta(t) = \frac{du(t)}{dt}$$ step function

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Drill Exercise 12.3

Evaluate the following integrals

(a) $I = \int_{-1}^{3} (t^3+2)\left[\delta(t) + 8\delta(t-1)\right] dt$

$= \int_{-1}^{3} (t^3+2)(\delta(t))dt + 8\int_{-1}^{3}(t^3+2)\delta(t-1)dt$

$= \int_{-1}^{3} (0^3+2)\,\delta(t)dt + 8\int_{-1}^{3}(1^3+2)\,\delta(t-1)dt$

$= 2 + 24 = \boxed{26}$

(b) $I = \int_{-2}^{2} t^2\left[\delta(t) + \delta(t+1.5) + \delta(t-3)\right] dt$

$I = \int_{-2}^{2} t^2\delta(t)dt + \int_{-2}^{2} t^2\delta(t+1.5)dt + \int_{-2}^{2} t^2\delta(t-3)dt$

$I = \int_{-2}^{2} 0^2\delta(t)dt + \int_{-2}^{2} (1.5)^2\delta(t+1.5)dt + \int_{-2}^{2} (3)^2\delta(t-3)dt$

$= 0\int_{-2}^{2}\delta(t)dt + 2.25\int_{-2}^{2}\delta(t+1.5)dt + 9\int_{-2}^{2}\delta(t-3)dt$

$= 0(1) + 2.25(1) + 9(0)$

$\boxed{I = 2.25}$

CIRCUITS II

Functional Transforms

- Define all functions to be zero for $t < 0^-$

$u(t) = 1$ for $t > 0$

Step Function

$$\mathcal{L}\{u(t)\} = \int_0^\infty u(t)\, e^{-st} = \int_{0^+}^\infty 1 e^{-st}\, dt$$

$$= \left.\frac{e^{-st}}{-s}\right|_{0^+}^{\infty} = 0 - \left(\frac{1}{-s}\right) = \boxed{\frac{1}{s}}$$

$$\mathcal{L}\{e^{-at}\} = \int_0^\infty e^{-at} e^{-st}\, dt = \int_0^\infty e^{-(a+s)t}\, dt$$

$$= \left.\frac{e^{-(a+s)t}}{-(a+s)}\right|_0^\infty = 0 - \frac{1}{-(a+s)}$$

$$\boxed{\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}}$$

Table 12.1 => Abbreviated List of Laplace Transforms

CIRCUITS II

Operational Transforms

$$\mathcal{L}\{f(t)\} = F(s)$$

Multiplication by a constant

$$\boxed{\mathcal{L}\{K f(t)\} = K F(s)}$$

Addition / Subtraction

If $\mathcal{L}\{f_1(t)\} = F_1(s)$, $\mathcal{L}\{f_2(t)\} = F_2(s)$,
$\mathcal{L}\{f_3(t)\} = F_3(s)$

$$\boxed{\mathcal{L}\{f_1(t) + f_2(t) - f_3(t)\} = F_1(s) + F_2(s) + F_3(s)}$$

Differentiation

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^-)$$

$$\mathcal{L}\left\{\frac{d^2f(t)}{dt^2}\right\} = s^2F(s) - sf(0^-) - \frac{df(0^-)}{dt}$$

$$\mathcal{L}\left\{\frac{d^nf(t)}{dt^n}\right\} = s^nF(s) - s^{n-1}f(0^-) - s^{n-2}\frac{df(0^-)}{dt} - s^{n-3}\frac{d^2f(0^-)}{dt^2} - \cdots - \frac{d^{n-1}f(0^-)}{dt^{n-1}}$$

Integration

$$\boxed{\mathcal{L}\left\{\int_{0^-}^{t} f(x)\,dx\right\} = \frac{F(s)}{s}}$$

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Translation in the time domain

$$\mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-as}F(s),$$

must be the same

Translation in the Frequency Domain

$$\boxed{\mathcal{L}\{e^{-at}f(t)\} = F(s+a)}$$

$$\mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}\{e^{-at}\sin\omega t\} = \frac{\omega}{(s+a)^2+\omega^2}$$

Scale Changing

$$\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right), \quad a>0$$

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$$

$$\mathcal{L}\{\sin\omega t\} = \frac{1}{\omega}\,\frac{1}{\left(\frac{s}{\omega}\right)^2+1} = \frac{1}{\omega}\left(\frac{1}{\frac{1}{\omega^2}(s^2+\omega^2)}\right)$$

$$\boxed{\mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2}}$$

Table 12.2 => List of Operational Transforms

Drill Exercise 12.6

1

Drill Exercise 12.6

Find the Laplace Transform of

(a) $t^2 e^{-at}$

$$\mathcal{L}\{e^{-at}\underbrace{f(t)}_{t^2}\} = F(s+a)$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}\{t^2\} = \frac{2}{s^3}$$

$$\boxed{\mathcal{L}\{t^2 e^{-at}\} = \frac{2}{(s+a)^3}}$$

(b) $\frac{d}{dt}\left(e^{-at}\sinh\beta t\right)$

$$\mathcal{L}\{\sinh\beta t\} = \frac{\beta}{s^2-\beta^2}$$

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^-)$$

$$F(s) = \frac{\beta}{(s+a)^2-\beta^2} \qquad f(0^-) = 0$$

$$\boxed{\mathcal{L}\left\{\frac{d}{dt}e^{-at}\sinh\beta t\right\} = \frac{s\beta}{(s+a)^2-\beta^2}}$$

(c) $t\cos\omega t$

$$12.2 \Rightarrow \mathcal{L}\{tf(t)\} = -\frac{dF(s)}{ds}$$

$f(t) = \cos\omega t$

$$F(s) = \frac{s}{s^2+\omega^2} \qquad \frac{dF(s)}{ds} = \frac{(s^2+\omega^2)(1)-(s)(2s)}{(s^2+\omega^2)^2}$$

$$\frac{dF(s)}{ds} = \frac{\omega^2-s^2}{(s^2+\omega^2)^2} \Rightarrow \mathcal{L}\{t\cos\omega t\} = -\frac{dF(s)}{ds} = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

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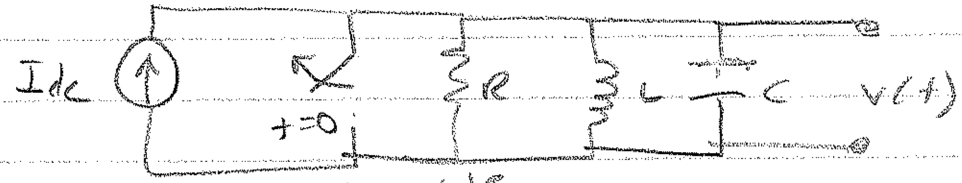
CIRCUITS II

Applying the Laplace Transform



dummy variable

KCL

$$\frac{v(t)}{R} + \frac{1}{L}\int_0^t v(x)\,dx + C\frac{dv(t)}{dt} = I_{dc}\,u(t)$$

Integrodifferential equation

Take the Laplace Transform of the integrodi~~ff~~fferential equation

$$\mathcal{L}\left\{\frac{v(t)}{R} + \frac{1}{L}\int_0^t v(x)\,dx + C\frac{dv(t)}{dt} = I_{dc}\,u(t)\right\}$$

$$\frac{1}{R}\mathcal{L}\{v(t)\} + \frac{1}{L}\mathcal{L}\left\{\int_0^t v(x)\,dx\right\} + C\,\mathcal{L}\left\{\frac{dv(t)}{dt}\right\} = I_{dc}\,\mathcal{L}\{u(t)\}$$

Let $\mathcal{L}\{v(t)\} = V(s)$

Assume = 0 initially

$$\frac{1}{R}V(s) + \frac{1}{L}\frac{V(s)}{s} + C\left[sV(s) - \cancel{v(0^-)}\right] = I_d\,\frac{1}{s}$$

$$V(s)\left(\frac{1}{R} + \frac{1}{sL} + Cs\right) = I_{dc}\frac{1}{s}$$

$$\boxed{V(s) = \frac{I_{dc}/C}{s^2 + (1/RC)s + (1/LC)}}$$

$v(t) \Rightarrow$ Take the inverse Transform

$$v(t) = \mathcal{L}^{-1}\{V(s)\}$$

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CIRCUITS II

Inverse Transforms

Need to get $V(s)$ in a form where we can take the inverse transform
$\Rightarrow$ Use partial fractions

$$F(s) = \frac{s+6}{s^2(s+1)(s+4)^2}$$

$$\frac{s+6}{s^2(s+1)(s+4)^2} = \frac{K_1}{s} + \frac{K_2}{s^2} + \frac{K_3}{s+1} + \frac{K_4}{s+4} + \frac{K_5}{(s+4)^2}$$

$$= K_1 + K_2 t + K_3 e^{-t} + K_4 e^{-4t} + K_5 t e^{-4t}$$

Just have to find $K_1 - K_5$

Tricks

$$F(s) = \frac{2}{s^2+25} = \frac{2}{s^2+5^2}$$

$$\mathcal{L}\{\sin 5t\} = \frac{5}{s^2+5^2}$$

$$F(s) = \frac{2}{5}\,\frac{5}{s^2+5^2}$$

$$\boxed{f(t) = \frac{2}{5}\sin 5t}$$

$$F(s) = \frac{2}{s^2+5s+6} = \frac{2}{(s+3)(s+2)} = \boxed{\frac{K_1}{s+3} + \frac{K_2}{s+2}}$$

$$F(s) = \frac{2}{s^2+2s+4} = \frac{2}{s^2+2s+1+3}$$

$$= \frac{2}{(s+1)^2+3} = \frac{2}{\sqrt{3}}\,\underbrace{\frac{\sqrt{3}}{(s+1)^2+(\sqrt{3})^2}}_{\mathcal{L}^{-1} = e^{-t}\sin(\sqrt{3}t)}$$

$$\boxed{f(t) = \frac{2}{\sqrt{3}}e^{-t}\sin(\sqrt{3}t)}$$

CIRCUITS II

Complex Roots

$$F(s) = \frac{(s+3)}{s^2+6s+25} \qquad s = \frac{-6 \pm \sqrt{6^2 - 4(25)}}{2} = -3 \pm j4$$

$$s^2+6s+25 = (s+3-j4)(s+3+j4)$$

$$\frac{s+3}{s^2+6s+25} = \frac{K}{s+3-j4} + \frac{K^*}{s+3+j4}$$

$$K = |K| \angle \theta^\circ, \quad K^* = |K| \angle -\theta^\circ$$

Finding the K's

$$\frac{s+4}{s^2+4s+3} = \frac{s+4}{(s+3)(s+1)} = \frac{K_1}{s+3} + \frac{K_2}{s+1}$$

K_1

- Find what value of s makes the denominator = 0

$s=-3$

- Substitute $s=-3$ into the left side excluding the denominator $s+3$

$$K_1 = \frac{s+4}{\cancel{(s+3)}(s+1)}\Bigg|_{s=-3} = \frac{-3+4}{-3+1} = \boxed{-\frac{1}{2}}$$

K_2

$$K_2 = \frac{s+4}{(s+3)\cancel{(s+1)}}\Bigg|_{s=-1} = \frac{-1+4}{-1+3} = \boxed{\frac{3}{2}}$$

This method does not work for repeated roots

- Modified for repeated roots

$$\frac{100(s+25)}{s(s+5)^2} = \frac{K_1}{s} + \frac{K_2}{(s+5)^2} + \frac{K_3}{(s+5)}$$

$$K_1 = \frac{100(s+25)}{\cancel{s}(s+5)^2}\Bigg|_{s=0} = \frac{250}{125} = 20$$

CIRCUITS II

$$K_2 = \left.\frac{100(s+25)}{s\cancel{(s+5)^2}}\right|_{s=-5} = \frac{100(-5+25)}{-5} = \boxed{-400}$$

K_3

$$\frac{100(s+25)}{s(s+5)^2} = \frac{K_1}{s} + \frac{K_2}{(s+5)^2} + \frac{K_3}{(s+5)}$$

Multiply both sides by $(s+5)^2$

$$\frac{100(s+25)}{s} = \frac{K_1}{s}(s+5)^2 + K_2 + K_3(s+5)$$

- Differentiate both sides with respect to s
- Substitute $s = -5$

1

Drill Exercise 12.10

$$F(s) = \frac{10(s^2+119)}{(s+5)(s^2+10s+169)}$$

Partial Fractions

$$\frac{10s^2+1190}{(s+5)(s^2+10s+169)} = \frac{K_1}{s+5} + \frac{K_2 s + K_3}{s^2+10s+169}$$

$$10s^2 + 1190 = K_1 s^2 + 10K_1 s + 169K_1 + K_2 s^2 + 5K_2 s + K_3 s + 5K_3$$

$$K_1 s^2 + K_2 s^2 = 10s^2 \qquad K_1 + K_2 = 10$$

$$10K_1 s + 5K_2 s + K_3 s = 0 \qquad K_1 + 5K_2 + K_3 = 0$$

$$169K_1 + 5K_3 = 0 \qquad 169K_1 + 5K_3 = 0$$

3 Equations, 3 Unknowns

$$K_1 = 10,\ K_2 = 0,\ K_3 = -100$$

$$F(s) = 10\left(\frac{1}{s+5}\right) - 100\left(\frac{1}{(s+5)^2+(12)^2}\right)$$

$$F(s) = 10\left(\frac{1}{s+5}\right) - \frac{100}{12}\left(\frac{12}{(s+5)^2+(12)^2}\right)$$

$$\boxed{f(t) = \left(10e^{-5t} - 8.33e^{-5t}\sin 12t\right)u(t)}$$

Alternate Method

$$s = \frac{-10 \pm \sqrt{100-4(169)}}{2} = -5 \pm j12$$

$$\frac{10(s^2+119)}{(s+5)(s^2+10s+169)} = \frac{K_1}{s+5} + \frac{K_2}{s+5-j12} + \frac{K_2^*}{s+5+j12}$$

$$K_1 = \left.\frac{10(s^2+119)}{\cancel{(s+5)}(s^2+10s+169)}\right|_{s=-5} = 10$$

②

Drill Exercise 12.10

$$K_2 = \frac{10(s^2+119)}{(s+5)\cancel{(s+5-j12)}(s+5+j12)}\Bigg|_{s=-5+j12}$$

$$= \frac{10\left[(-5+j12)^2+119\right]}{(-5+j12+5)(-5+j12+j12)}$$

$$K_2 = j4.17 = 4.17\angle 90^\circ$$

$$K_2^* = 4.17\angle -90^\circ$$

Theorem for complex roots

For $F(s) = \frac{K}{s+\alpha-j\beta} + \frac{K^*}{s+\alpha+j\beta}$ where $K = |K|\angle\theta$

$$\boxed{f(t) = 2|K|e^{-\alpha t}\cos(\beta t+\theta)\,u(t)}$$

$$\boxed{f(t) = \left[10e^{-5t} + 2(4.17)e^{-5t}\cos(12t+90^\circ)\right]u(t)}$$

$$\boxed{f(t) = \left[10e^{-5t} - 8.33e^{-5t}\sin 12t\right]u(t)}$$

CIRCUITS II

Initial and Final Value Theorems

Determine the behavior of $f(t)$ at $t=0$ & $t=\infty$ using $F(s)$

Initial-Value Theorem

$$\boxed{\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)}$$

Final-Value Theorem

$$\boxed{\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)}$$

Drill Exercise 12.16

(1)

Drill Exercise 12.16

Find the initial and final values of f(t) for

$$F(s) = \frac{7s^2+63s+134}{(s+3)(s+4)(s+5)}$$

Initial Value

$\lim_{s\to\infty} sF(s)$

$$\lim_{s\to\infty} s\,\frac{7s^2+63s+134}{(s+3)(s+4)(s+5)}$$

$$= s\,\frac{7s^2(1+9/s+134/7s^2)}{s^3(1+3/s)\cdot(1+4/s)(1+5/s)}$$

$$= \frac{7(1+0+0)}{(1+0)(1+0)(1+0)} = 7$$

$$\boxed{f(0^+) = 7}$$

Final Value

$$\lim_{s\to 0} sF(s) = \frac{7s^3+63s^2+134s}{(s+3)(s+4)(s+5)} = \frac{0}{60} = 0$$

$$\boxed{f(\infty) = 0}$$