CIRCUITS II

Chapter 13: Laplace Transform in Circuit Analysis

Circuit Elements in the s Domain

$f(t) \Rightarrow$ time domain $\quad v(t), i(t)$

$F(s) \Rightarrow \quad$ s – Domain $\quad V(t), I(t)$

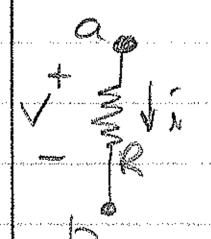
$V = \mathcal{L}\{v\}, \; I = \mathcal{L}\{i\}$

Steps

1) Write the time-domain equation relating voltage to current for the element
$\Rightarrow$ s-domain current to voltage relation
2) Construct a circuit model that satisfies the s-domain current to voltage relation

Note: voltage in time-domain $\Rightarrow$ volts; s-domain, volt-sec

Resistor in the s Domain



$v(t) = R\,i(t)$

$\boxed{v = Ri}$

Time Domain

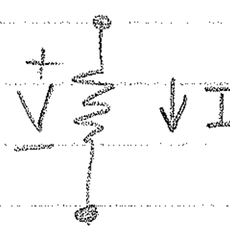
$V = \mathcal{L}\{v\}$

$= \mathcal{L}\{iR\}$

$V = R\,\mathcal{L}\{i\}$

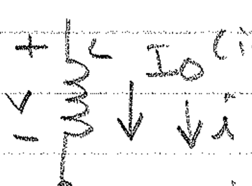
$\boxed{V = RI}$

$\boxed{V(s) = R\,I(s)}$



s-Domain

Inductor in the s Domain



time domain

$v(t) = L\,\frac{di(t)}{dt}$

$V(s) = L[\,sI(s) - i(0^-)\,]$

$V(s) = sL\,I(s) - L\,I_0$ or $V = sLI - LI_0$

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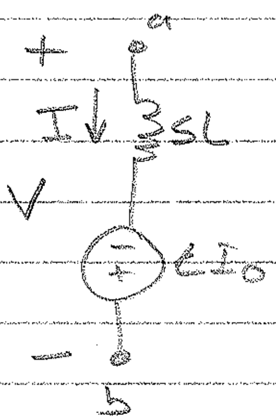
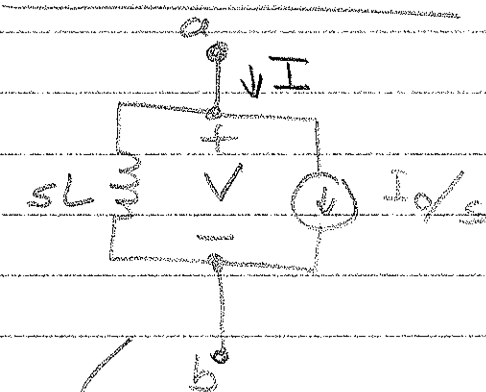
Circuit Model

Series

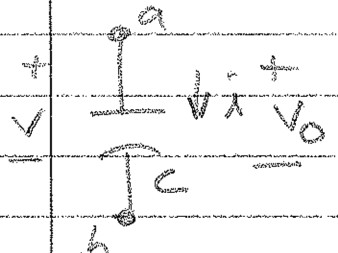


Parallel Model



$$\boxed{I = \frac{V + LI_0}{sL} = \frac{V}{sL} + \frac{I_0}{s}}$$

Capacitor in the s Domain



$$i = C\frac{dv}{dt}$$

$$I = C\left[sV - v(0^-)\right]$$

$$\boxed{I = sCV - v(0^-)}$$

$$V = \frac{1}{C}\int_{0^-}^{t} i\, d\tau + V_0$$

$$= \frac{1}{C}\left[\frac{I(s)}{s}\right] + \frac{V_0}{s}$$

$$\boxed{V = \frac{I}{sC} + \frac{V_0}{s}}$$

CIRCUITS II

Circuit Analysis in the s Domain

$V = ZI$ (Assuming no initial energy is stored in the inductor or capacitor)

$Z \Rightarrow$ s domain impedance

Resistor $\Rightarrow$ R ohms

Inductor $\Rightarrow$ sL ohms

Capacitor $\Rightarrow$ $1/sC$ ohms

Rules for s-domain impedances are the same as for the frequency domain impedances

- Series to parallel
- $\Delta - Y$ transformation
- KVL, KCL

Drill Exercise 13.1

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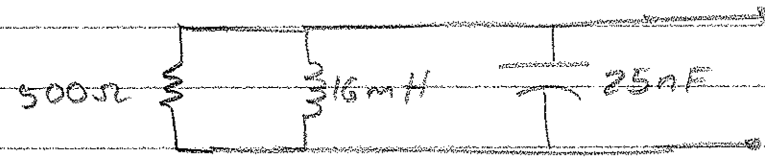
Drill Exercise 13.1



(a) Express the admittance of this parallel combination as a rational function of s

$$Y = \frac{1}{Z} \qquad Y_{eq} = \sum \frac{1}{Z_{eq}}$$

$$Y = \frac{1}{R} + \frac{1}{sL} + sC$$

$$= C\frac{1}{s}\left(s^2 + \frac{1}{RC}s + 1/LC\right)$$

$$Y = \frac{C\left[s^2 + \left(\frac{1}{RC}\right)s + \left(1/LC\right)\right]}{s}$$

$$\frac{1}{RC} = \frac{1}{(500)(25\times10^{-9})} = 80{,}000$$

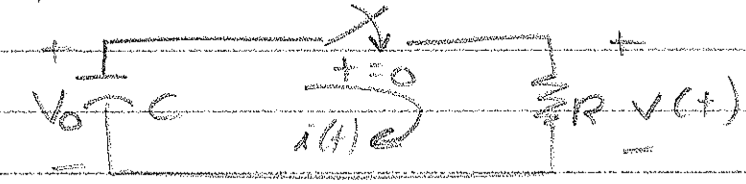
$$\frac{1}{LC} = \frac{1}{(16\times10^{-3})(25\times10^{-9})} = 25\times10^{8}$$

$$\boxed{Y = \frac{25\times10^{-9}\left(s^2 + 80{,}000s + 25\times10^{8}\right)}{s}}$$

CIRCUITS II

Circuit Applications of the Laplace Transform

Natural Response of an RC Circuit



KVL

$$-V_c(t) + v(t) = 0$$

$$-\left(\frac{-1}{C}\int_0^t i(t)\,d\tau + V_0\right) + i(t)R = 0$$

s-Domain

$$-\left(\frac{-I(s)}{sC} + \frac{V_0}{s}\right) + I(s)R = 0$$

$$I(s)\left(\frac{1}{sC} + R\right) = \frac{V_0}{s}$$

$$I(s)(1 + sRC) = V_0 C$$

$$I(s) = \frac{V_0 C}{(1 + sRC)} = \frac{V_0 C}{RC\left(\frac{1}{RC} + s\right)}$$

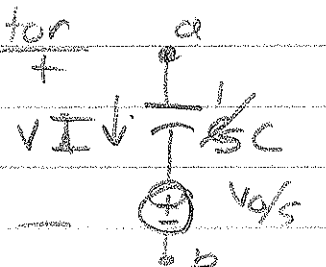
$$I(s) = \frac{V_0/R}{(s + 1/RC)}$$

$$\boxed{i(t) = \frac{V_0}{R} e^{-t/RC} u(t)}$$

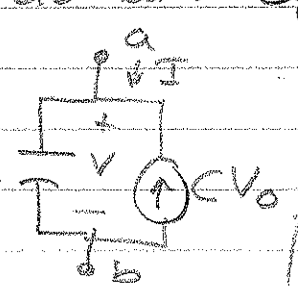
$$\boxed{v = Ri = V_0 e^{-t/RC} u(t)}$$

Could also use the s-domain equivalent

Capacitor



$$V = \frac{I}{sC} + \frac{V_0}{s}$$

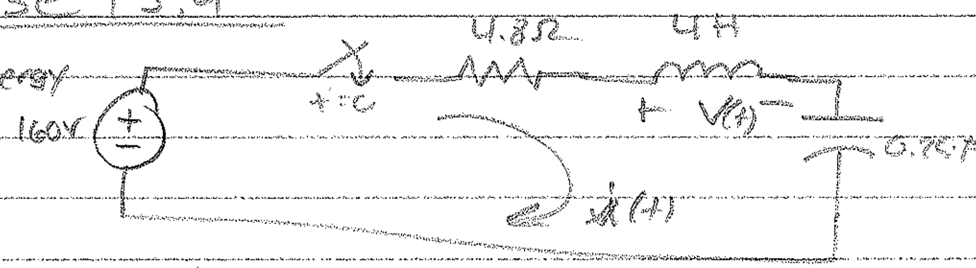


$$\frac{1}{sC}I(s) - \frac{V_0}{s} + RI = 0$$

Drill Exercise 13.4

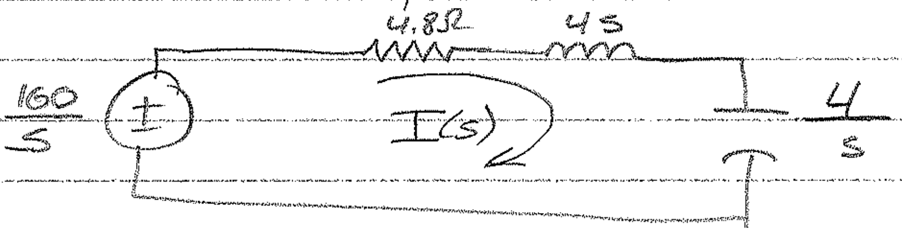
Drill Exercise 13.4

No initial energy is stored



(a) Find the s-domain expression for I

s-domain



$$\frac{160}{s} = 4.8\,I(s) + 4s\,I(s) + \frac{4}{s}\,I(s)$$

$$160 = 4.8s\,I(s) + 4s^2 I(s) + 4\,I(s)$$

$$I(s) = \frac{160}{4s^2 + 4.8s + 4}$$

$$\boxed{I(s) = \frac{40}{s^2 + 1.2s + 1}}$$

(b) Find $i(t)$

$$s = \frac{-1.2 \pm \sqrt{(1.2)^2 - (4)(1)}}{2} = -0.6 \pm j0.8$$

$$\frac{40}{s^2+1.2s+1} = \frac{K}{s+0.6-j0.8} + \frac{K^*}{s+0.6+j0.8}$$

$$K = \left.\frac{40}{\cancel{(s+0.6-j0.8)}(s+0.6+j0.8)}\right|_{s=-0.6+j0.8} = \frac{40}{j1.6}$$

$$K = -j25 = 25\angle{-90^\circ} \qquad K^* = 25\angle{90^\circ}$$

$$i(t) = 2(25)e^{-0.6t}\cos(0.8t - 90^\circ)\,u(t) \text{ A}$$

$$\boxed{i(t) = 50e^{-0.6t}\sin 0.8t)\,u(t) \text{ A}}$$

2

Drill Exercise 13.4

(c) Find $V(s)$

$$V(s) = Z\,I(s)$$

$$V(s) = \frac{(4s)(40)}{s^2 + 1.2s + 1}$$

$$\boxed{V(s) = \frac{160s}{s^2 + 1.2s + 1}}$$

(d) Find $v(t)$

$$\frac{160s}{(s + 0.6 - j0.8)(s + 0.6 + j0.8)} = \frac{K}{s + 0.6 - j0.8} + \frac{K^*}{s + 0.6 + j0.8}$$

$$K = \left.\frac{160s}{\cancel{(s + 0.6 - j0.8)}(s + 0.6 + j0.8)}\right|_{s = -0.6 + j0.8}$$

$$= \frac{160(-0.6 + j0.8)}{j1.6} = \frac{-160(-j0.6 - 0.8)}{1.6}$$

$$K = 80 + j60 = 100\angle 36.87^\circ$$

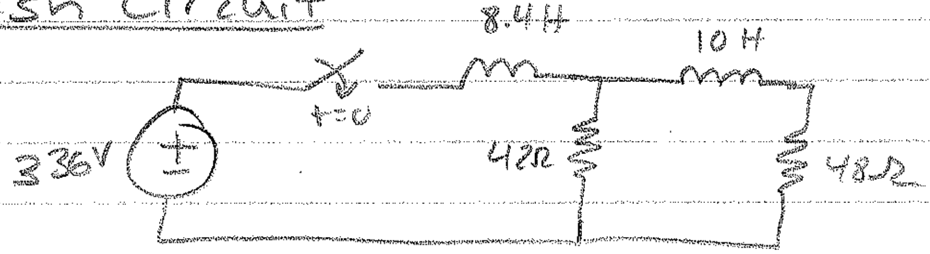
$$v(t) = 2(100)e^{-0.6t}\cos(0.8t + 36.87^\circ)\,u(t) \text{ V}$$

$$\boxed{v(t) = \left[200e^{-0.6t}\cos(0.8t + 36.87^\circ)\right]u(t) \text{ V}}$$

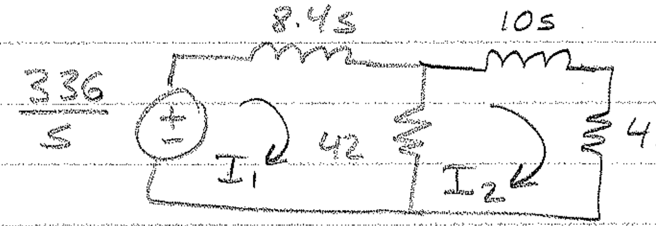
(32)

CIRCUITS II

Multiple Mesh Circuit



S-Domain



Mesh Current

$$-\frac{336}{s} + 8.4s I_1 + 42(I_1 - I_2) = 0$$

$$10s\, I_2 + 48 I_2 + 42(I_2 - I_1) = 0$$

$$(42 + 8.4s) I_1 - 42 I_2 = \frac{336}{s}$$

$$-42 I_1 + (90 + 10s) I_2 = 0$$

Cramer's Method

$$\Delta = \begin{vmatrix} 42 + 8.4s & -42 \\ -42 & 90 + 10s \end{vmatrix}$$

$$= (42 + 8.4s)(90 + 10s) - (-42)(-42)$$

$$= 3780 + 1176s + 84s^2 - 1764$$

$$= 84\left[s^2 + 14 + 24\right]$$

$$\Delta = 84(s+2)(s+12)$$

$$N_1 = \begin{vmatrix} 336/s & -42 \\ 0 & 90 + 10s \end{vmatrix} = \frac{336}{s}(90 + 10s)$$

$$N_1 = \frac{3360(s+9)}{s}$$

(33)

CIRCUITS II

$$N_2 = \begin{vmatrix} 42+8.4s & 336/s \\ -42 & 0 \end{vmatrix} = \frac{14,112}{s}$$

$$I_1 = \frac{N_1}{\Delta} = \frac{3360(s+9)}{s}\left(\frac{1}{84(s+2)(s+12)}\right)$$

$$\boxed{I_1 = \frac{40(s+9)}{s(s+2)(s+12)}}$$

$$I_2 = \frac{N_2}{\Delta} = \frac{14,112}{s}\left(\frac{1}{84(s+2)(s+12)}\right)$$

$$\boxed{I_2 = \frac{168}{s(s+2)(s+12)}}$$

CIRCUITS II

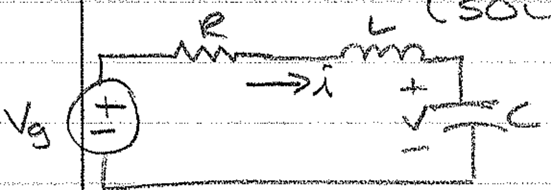
The Transfer Function

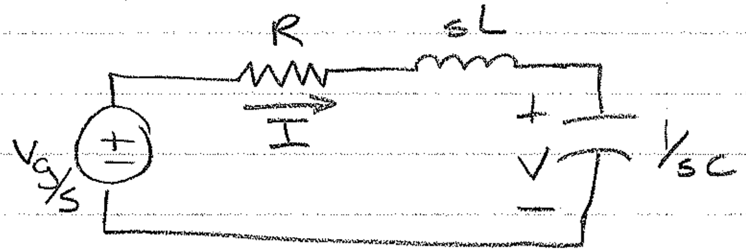
Transfer Function $\boxed{H(s) = \frac{Y(s)}{X(s)}}$ (Assume all initial conditions are zero)

$H(s) \Rightarrow$ Transfer Function

$Y(s) \Rightarrow$ Laplace Transform of the output (response)

$X(s) \Rightarrow$ Laplace Transform of the input (source)





Transfer Function for the current

$$I(s) = \frac{V_g/s}{R + sL + (1/sC)}$$

$$V_g(s) = \frac{V_g}{s}$$

$$H(s) = \frac{I(s)}{V_g(s)} = \frac{(V_g/s)}{(V_g/s)(R + sL + 1/sC)} = \frac{1}{R + sL + 1/sC} = \frac{sC}{s^2LC + RCs + 1}$$

Transfer Function for V

$$V(s) = I\left(\frac{1}{sC}\right) = \frac{V_g/s^2C}{R + sL + (1/sC)}$$

$$H(s) = \frac{V(s)}{V_g(s)} = \frac{(V_g/s)(1/sC)}{(V_g/s)(R + sL + 1/sC)} = \frac{(1/sC)}{(1/sC)(s^2LC + RCs + 1)}$$

$$\boxed{H(s) = \frac{1}{s^2LC + RCs + 1}}$$

Drill Exercise 13.9

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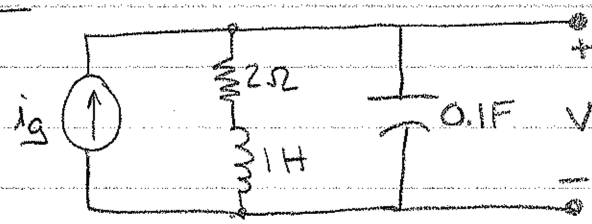
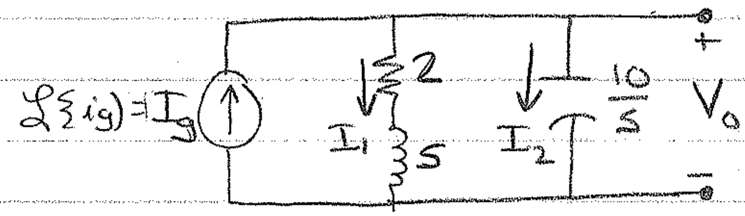
Drill Exercise 13.9



(a) Derive the numerical expression for the transfer function V_o/I_g



$$I_g = I_1 + I_2$$

$$I_g = \frac{V_o}{s+2} + \frac{V_o}{10/s}$$

$$V_o\left(\frac{s}{10} + \frac{1}{s+2}\right) = I_g$$

$$H(s)\ \frac{V_o}{I_g} = \frac{1}{\left(\frac{s}{10} + \frac{1}{s+2}\right)} = \frac{10(s+2)}{(s(s+2)+10)}$$

$$\boxed{H(s) = \frac{10(s+2)}{s^2+2s+10}}$$

(b) Find the numerical value of each pole and the zero of $H(s)$

zeros => value of s that makes $H(s)=0$

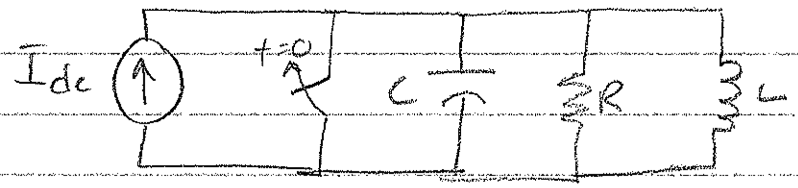
$= -2$

Poles => value of s that makes $H(s) \Rightarrow \infty$

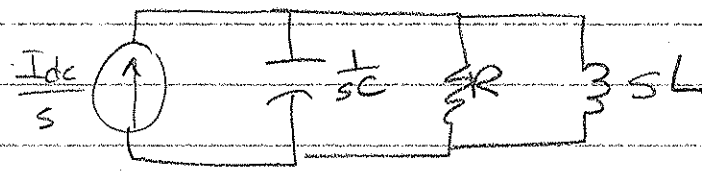
$$s = \frac{-2 \pm \sqrt{4-40}}{2} = \boxed{-1 \pm j3}$$

Alternating Current

No energy is stored

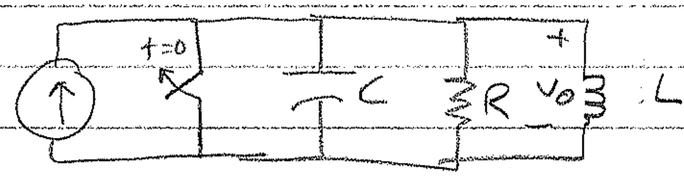


S-domain



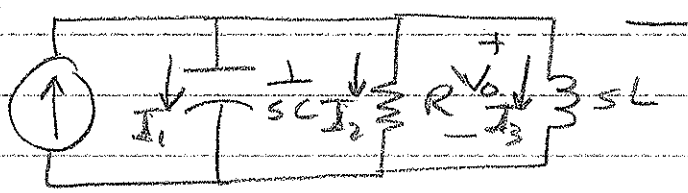
Alternating Current

$i = I_g \cos \omega t$



S-Domain

$$I = \frac{I_g s}{s^2 + \omega^2}$$



$$I = I_1 + I_2 + I_3$$

$$\boxed{\frac{I_g s}{s^2+\omega^2} = \frac{V_o}{(1/sC)} + \frac{V_o}{R} + \frac{V_o}{sL}}$$

Simplify + take the inverse Laplace Transform

Two terms => transient + steady-state

CIRCUITS II

(13.7) Transfer Function in Steady-State Sinusoidal Response

Can use the transfer function to get the steady-state sinusoidal response of a circuit

Given

Source $x(t) = A\cos(\omega t + \phi)$

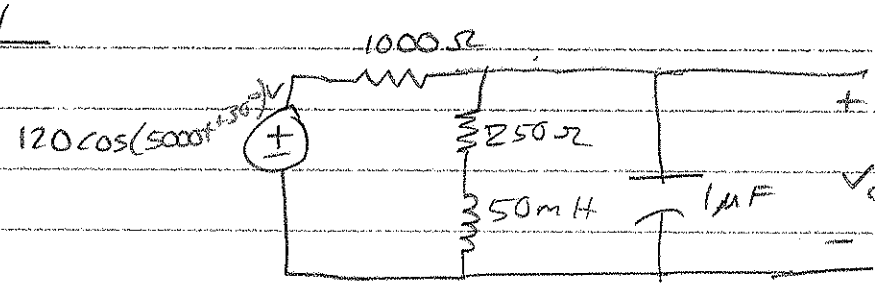
$H(s)$

Evaluate $H(s)$ at $j\omega$

$$H(j\omega) = |H(j\omega)| \angle \theta(\omega)^\circ$$

$$y_{ss}(t) = A|H(j\omega)|\cos[\omega t + \phi + \theta(\omega)]$$

Example 13.4



Transfer Function for v_o => $H(s) = \dfrac{1000(s+5000)}{s^2 + 6000s + 25\times10^6}$

Find $v_o(t)$ (steady-state)

$$H(j\omega) = H(j5000) = \frac{1000(j5000+5000)}{(j5000)^2 + (6000)(j5000) + 25\times10^6}$$

$$= \frac{1+j1}{j6} = \frac{1}{6} - j\frac{1}{6} = \frac{\sqrt{2}}{6}\angle -45^\circ$$

$$v_{o_{ss}}(t) = (120)\left(\frac{\sqrt{2}}{6}\right)\cos(5000t + 30^\circ + (-45^\circ)) \checkmark$$

$$\boxed{v_{o_{ss}}(t) = 20\sqrt{2}\cos(5000t - 15^\circ)\,V}$$