CIRCUITS II

Chapter 16: Fourier Series

Periodic Function => Repeats itself every T seconds, where T is the peroid

$$f(t) = f(t \pm nT)$$

Fourier => found that a periodic function can be represented by an infinite sum of sine and cosine functions

$$f(t) = a_v + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right)$$

where $n = 1, 2, 3, \ldots$

a_v, a_n, b_n => Fourier Coefficients

$\omega_0 = \frac{2\pi}{T}$ => Fundamental Frequency

Multiples of ω_0, $2\omega_0$, $3\omega_0$, $4\omega_0 \ldots$ => harmonic frequencies

Conditions for a convergent Fourier series

1) $f(t)$ must be single-valued
2) $f(t)$ must have a finite number of discontinuities in the periodic interval
3) $f(t)$ must have a finite number of maxima & minima in the periodic interval
4) $\int_{t_0}^{t_0+T} |f(t)|\,dt$ must exist

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Determining the Fourier Coefficients

$$a_v = \frac{1}{T}\int_{t_o}^{t_o+T} f(t)\,dt$$ } average value of $f(t)$

$$a_K = \frac{2}{T}\int_{t_o}^{t_o+T} f(t)\cos(K\omega_o t)\,dt$$

$$b_K = \frac{2}{T}\int_{t_o}^{t_o+T} f(t)\sin(K\omega_o t)\,dt$$

Drill Exercise 16.1

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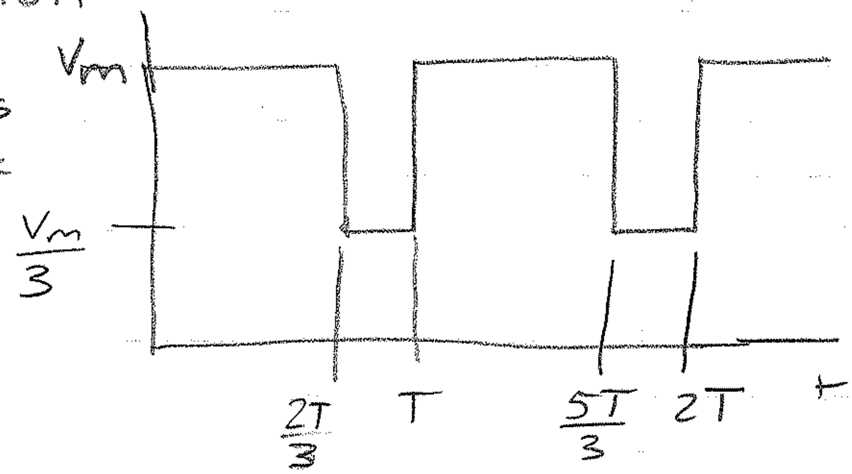
Drill Exercise 16.1

Derive expressions for a_v, a_k, and b_k if $V_m = 9\pi$ V



$$a_v = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\,dt$$

$$= \frac{1}{T}\int_0^T f(t)\,dt$$

$$= \frac{1}{T}\int_0^{2T/3} V_m\,dt + \frac{1}{T}\int_{2T/3}^{T} \frac{V_m}{3}\,dt$$

$$= \frac{9\pi}{T}\left(\frac{2T}{3} - 0\right) + \frac{9\pi}{3T}\left(T - \frac{2T}{3}\right)$$

$$= 6\pi + 3\pi - 2\pi = 7\pi = \boxed{21.99V}$$

$$a_K = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos(k\omega_0 t)$$

$$= \frac{2}{T}\left\{\int_0^{\frac{2T}{3}} V_m\cos(k\omega_0 t) + \int_{\frac{2T}{3}}^{T}\frac{V_m}{3}\cos(k\omega_0 t)\right\}$$

$$= \frac{2}{T}\left\{\frac{9\pi}{k\omega_0}\sin k\omega_0 t\Big|_0^{\frac{2T}{3}} + \frac{3\pi}{k\omega_0}\sin(k\omega_0 t)\Big|_{\frac{2T}{3}}^{T}\right\}$$

$$= \frac{6\pi}{k\omega_0 T}\left\{3\sin\left(k\omega_0\left(\frac{2T}{3}\right)\right) + \sin(k\omega_0 T) - \sin\left(k\omega_0\frac{2T}{3}\right)\right\}$$

$$a_k = \frac{6\pi}{k\omega_0 T}\left\{2\sin\left(\frac{2k\omega_0 T}{3}\right) + \sin(k\omega_0 T)\right\}$$

Drill Exercise 16.1

$$\omega_0 = \frac{2\pi}{T}$$

$$a_K = \frac{6\pi}{K\left(\frac{2\pi}{T}\right)(T)}\left\{ 2\sin\left(\frac{2K\left(\frac{2\pi}{T}\right)T}{3}\right) + \sin\left(K\left(\frac{2\pi}{T}\right)t\right)\right\}$$

$$a_K = \frac{3}{K}\left\{ 2\sin\left(\frac{4K\pi}{3}\right) + \cancelto{0}{\sin(2K\pi)}\right\}$$

$$\boxed{a_K = \frac{6}{K}\sin\left(\frac{4K\pi}{3}\right)}$$

$$b_K = \frac{2}{T}\left\{ \int_0^{\frac{2T}{3}} V_m \sin(k\omega_0 t)\,dt + \int_{\frac{2T}{3}}^{T} \frac{V_m}{3}\sin(k\omega_0 t)\,dt \right\}$$

$$= \frac{2}{T}\left\{ -\frac{9\pi}{k\omega_0}\cos(k\omega_0 t)\Big|_0^{\frac{2T}{3}} - \frac{3\pi}{k\omega_0}\cos(k\omega_0 t)\Big|_{\frac{2T}{3}}^{T}\right.$$

$$= -\frac{6\pi}{Tk\omega_0}\left\{ 3\cos\left(k\omega_0\left(\frac{2T}{3}\right)\right) - 3(1) + \cos(k\omega_0 T) - \cos\left(k\omega_0\frac{2T}{3}\right)\right\}$$

$$= -\frac{6\pi}{Tk\left(\frac{2\pi}{T}\right)}\left\{ -3 + 2\cos\left(k\left(\frac{2\pi}{T}\right)\left(\frac{2T}{3}\right)\right) + \cos\left(k\left(\frac{2\pi}{T}\right)T\right)\right.$$

$$= -\frac{3}{K}\left\{ -3 + 2\cos\left(\frac{4\pi k}{3}\right) + \underbrace{\cos(2\pi k)}_{1}\right\}$$

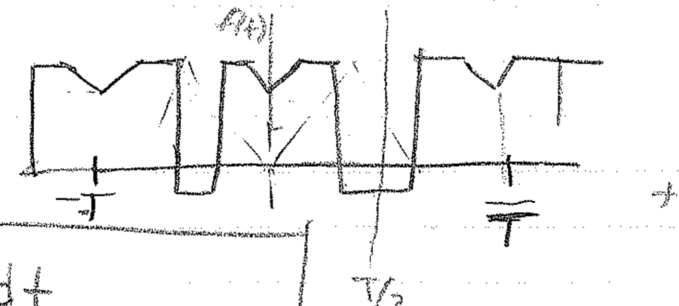
$$= -\frac{3}{K}\left\{ -2 + 2\cos\left(\frac{4\pi k}{3}\right)\right\}$$

$$\boxed{b_K = \frac{6}{K}\left(1 - \cos\left(\frac{4\pi k}{3}\right)\right)}$$

Effect of Symmetry on the Fourier Coefficient

Even Function

$f(t) = f(-t)$



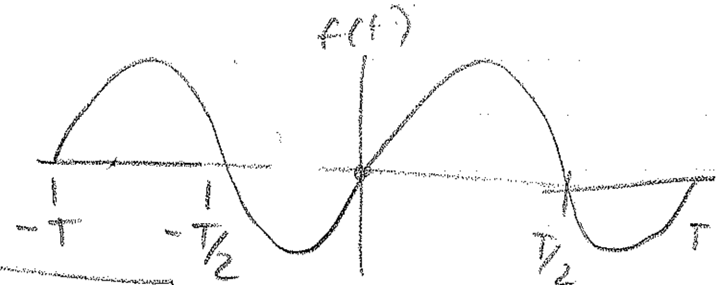
$$a_v = \frac{2}{T}\int_0^{T/2} f(t)\,dt$$

$$a_k = \frac{4}{T}\int_0^{T/2} f(t)\cos(k\omega_0 t)\,dt$$

$$b_k = 0 \quad \text{for all } k$$

Odd - Function

$f(t) = -f(-t)$



$$a_v = 0$$

$$a_k = 0 \quad \text{for all } k$$

$$b_k = \frac{4}{T}\int_0^{T/2} f(t)\sin(k\omega_0 t)\,dt$$

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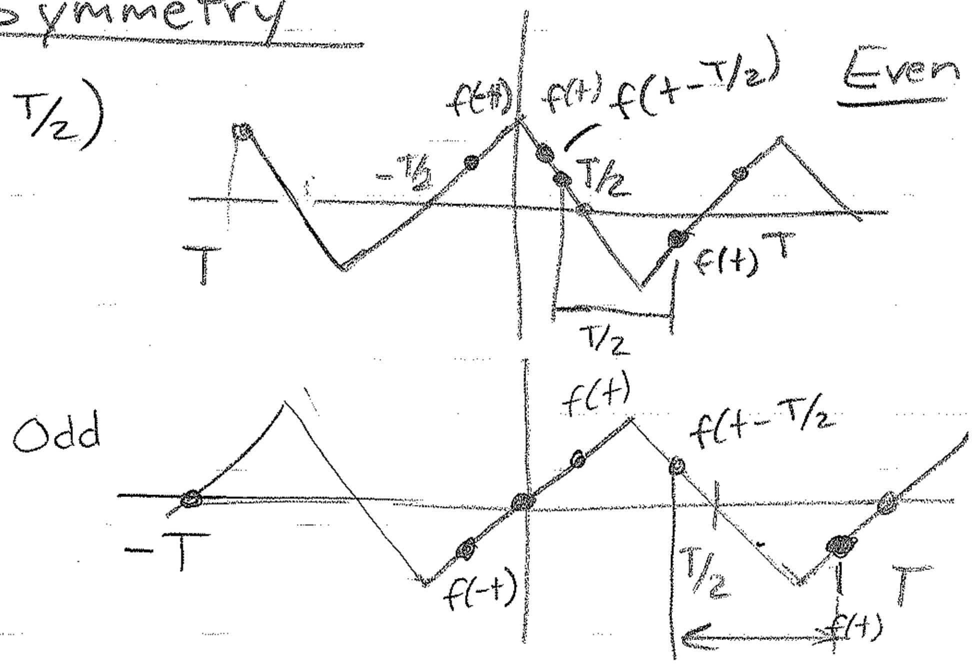
Half-Wave Symmetry

$f(t) = -f(t - T/2)$



$a_v = 0$

$a_K = 0$ for K even

$a_K = \frac{4}{T}\int_0^{T/2} f(t)\cos(K\omega_0 t)\,dt$ for K odd

$b_K = 0$ for K even

$b_K = \frac{4}{T}\int_0^{T/2} f(t)\sin(K\omega_0 t)\,dt$ for K odd

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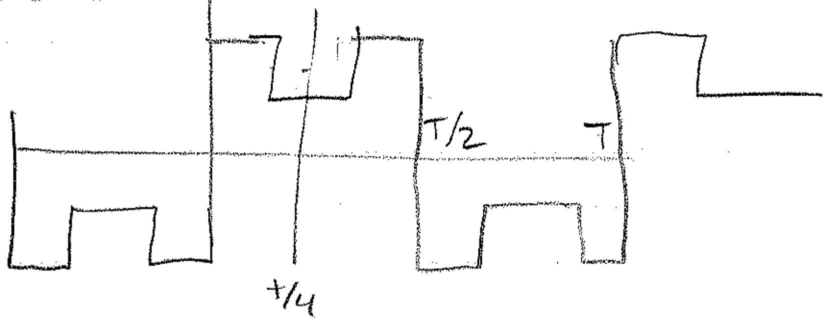
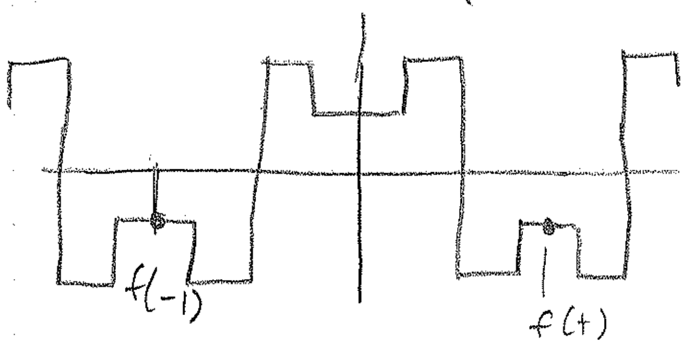
Quarter-Wave Symmetry

Half-Wave Symmetry & symmetry about the half-cycles
Can be made even or odd



Odd



Even

If Odd Function

$a_v = 0$ (half)

$a_K = 0$ for all K (odd)

$b_K = 0$ for K even

$$b_K = \frac{8}{T}\int_0^{T/4} f(t)\sin(k\omega_0 t)\,dt \quad \text{for K odd}$$

If Even Function

$a_v = 0$

$a_K = 0$ for K even

$$a_K = \frac{8}{T}\int_0^{T/4} f(t)\cos(K\omega_0 t)\,dt$$

$b_K = 0$ for all K

Drill Exercise 16.3

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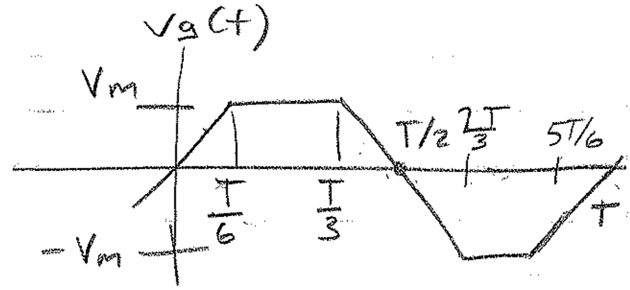
Drill Exercise 16.3

Derive the Fourier Series for the periodic voltage, v_g



Even or Odd

Half-Wave Yes

Quarter-Wave? Yes

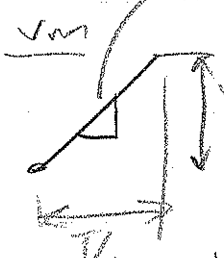
$a_v = 0$ for all K

$a_K = 0$ for all K

$b_K = 0$ for K even

$\frac{V_m}{T/6} = \frac{6V_m(t)}{T}$

$$b_K = \frac{8}{T}\int_0^{T/4} f(t)\sin(K\omega_0 t) \quad \text{for K odd}$$



$$= \frac{8}{T}\left\{ \int_0^{T/6}\left(\frac{6V_m}{T}\right)t\sin(k\omega_0 t)\,dt + \int_{T/6}^{T/4} V_m \sin(k\omega_0 t)\,dt \right\}$$

$$= \frac{8}{T}\left\{ \frac{6V_m}{T}\left(\frac{1}{(K\omega_0)^2}\sin(K\omega_0 t) - \frac{t}{K\omega_0}\cos(K\omega_0 t)\right)\Bigg|_0^{T/6} - \frac{1}{K\omega_0}\left(V_m\cos(K\omega_0 t)\right)\Bigg|_{T/6}^{T/4} \right\}$$

$$= \frac{8}{T}\left\{ \frac{6V_m}{TK\omega_0}\left[\frac{1}{K\omega_0}\left(\sin\left(\frac{K\omega_0 T}{6}\right)\right) - \cancel{\frac{T}{6}\cos\left(K\omega_0\frac{T}{6}\right)}\right] - \frac{T}{6}\cos\left(K\omega_0\frac{T}{4}\right) + \cancel{\frac{T}{6}\cos\left(K\omega_0\frac{T}{6}\right)} \right\}$$

$$= \frac{8}{T}\left\{ \frac{6V_m}{TK\omega_0}\left(\frac{1}{K\omega_0}\sin\left(\frac{K\omega_0 T}{6}\right) - \frac{T}{6}\cos\left(\frac{K\omega_0 T}{4}\right)\right)\right\}$$

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Drill Exercise 16.3

$$b_K = \frac{48V_m}{T^2K^2\omega_0^2}\sin\left(\frac{K\omega_0 T}{6}\right) - \frac{8V_m}{TK\omega_0}\cos\left(\frac{K\omega_0 T}{4}\right)$$

$$\omega_0 = \frac{2\pi}{T}$$

$$b_K = \frac{48V_m}{T^2K^2\left(\frac{2\pi}{T}\right)^2}\sin\left(\frac{KT}{6}\left(\frac{2\pi}{T}\right)\right) - \frac{8V_m}{TK\left(\frac{2\pi}{T}\right)}\cos\left(\frac{KT}{4}\left(\frac{2\pi}{T}\right)\right)$$

$$b_K = \frac{12V_m}{K^2\pi^2}\sin\left(\frac{K\pi}{3}\right) \underbrace{- \frac{4V_m}{K\pi}\cos\left(\frac{K\pi}{2}\right)}_{\text{for K odd } \cos\left(\frac{K\pi}{2}\right) = 0}$$

$$b_K = \frac{12V_m}{K^2\pi^2}\sin\left(\frac{K\pi}{3}\right)$$

$$V_g(t) = a_v + \sum_{n=1}^{\infty}\left(a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)\right)$$

$$V_g(t) = \sum_{n=1,3,5,\ldots}^{\infty}\frac{12V_m}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right)\sin(n\omega_0 t)$$

$$\boxed{V_g(t) = \frac{12V_m}{\pi^2}\sum_{\substack{n=1,3,5,\ldots\\ n=\text{odd}}}^{\infty}\frac{\sin(n\pi/3)}{n^2}\sin(n\omega_0 t)}$$

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Alternative Trigonometric Form of the Fourier Series

$$f(t) = a_v + \sum_{n=1}^{\infty}\left(a_n\cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right)$$

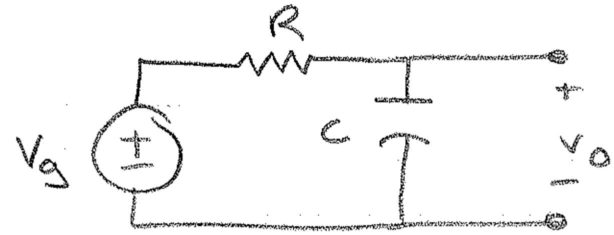
$$= a_v = \sum_{n=1}^{\infty}\left[\underbrace{a_n\cos(n\omega_0 t)} + \underbrace{b_n\cos(n\omega_0 t - 90^\circ)}\right]$$

Phasors => $a_n \angle 0^\circ$ $\qquad b_n \angle -90^\circ = -jb_n$

$$a_n - jb_n = \sqrt{a_n^2+b_n^2}\ \angle -\theta_n$$
$$= A_n \angle -\theta_n$$

$$\boxed{f(t) = a_v + \sum_{n=1}^{\infty} A_n\cos(n\omega_0 t - \theta_n)}$$

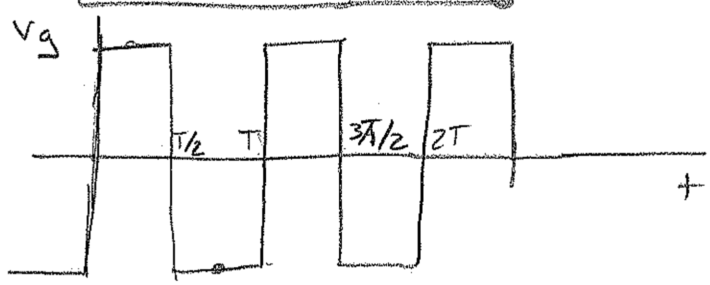
Application



Even or Odd

Half-Wave? Yes

Quarter Wave? Yes



$a_v = 0$

$a_k = 0$ for all k

$$b_k = \frac{8}{T}\int_0^{T/4} V_m \sin(k\omega_0 t)\,dt \quad \text{for } k \text{ odd}$$

$$= \frac{-8V_m}{Tk\omega_0}\cos(k\omega_0 t)\Big|_0^{T/4}$$

$$= \frac{-8V_m}{Tk\omega_0}\left[\cos\left(k\omega_0 \frac{T}{4}\right) - 1\right]$$

$$= \frac{-8V_m}{Tk\left(\frac{2\pi}{T}\right)}\left[\cos\left(k\left(\frac{2\pi}{T}\right)\left(\frac{T}{4}\right)\right) - 1\right]$$

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$$b_k = \frac{-4V_m}{k\pi}\left(\underbrace{\cos\left(k\frac{\pi}{2}\right)}_{=0 \text{ for } k \text{ odd}} - 1\right)$$

$$b_k = \frac{4V_m}{k\pi}$$

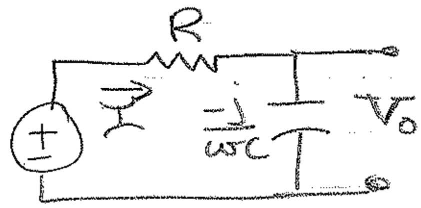
$$V_g = \frac{4V_m}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n}\sin(n\omega_0 t)$$

Expand the series

$$V_g = \frac{4V_m}{\pi}\sin(\omega_0 t) + \frac{4V_m}{3\pi}\sin(3\omega_0 t) + \frac{4V_m}{5\pi}\sin(5\omega_0 t) + \frac{4V_m}{7\pi}\sin(7\omega_0 t) + \ldots$$

Voltage Source $\overset{(V_g)}{\Rightarrow}$ infinitely many series-connected sinusoidal sources.

$$-\bar{V}_g + \bar{I}\left(R - \frac{j}{\omega C}\right) = 0$$



$$\bar{I} = \frac{\bar{V}_g}{R - \frac{j}{\omega C}}$$

$$\bar{V}_o = \bar{I}\left(\frac{-j}{\omega C}\right)$$

$$\bar{V}_o = \frac{-j\,\bar{V}_g}{\omega C\left(R - \frac{j}{\omega C}\right)} = \frac{-j\,\bar{V}_g}{\omega RC - j}\left(\frac{j}{j}\right)$$

$$\boxed{\bar{V}_o = \frac{\bar{V}_g}{1 + j\omega RC}}$$ for each source.

Since sources are sine functions, use phasors in terms of sine instead of cosine

For the fundamental frequency $(n=1)$

$$\bar{V}_{o1} = \frac{\frac{4V_m}{\pi}\angle 0^\circ}{1 + j\omega_0 RC} = \frac{4V_m}{\pi\sqrt{1+\omega_0^2R^2C^2}\angle B_1}$$

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$$\beta_1 = \tan^{-1}(\omega_0 RC)$$

$$\bar{V}_{o1} = \frac{4V_m}{\pi\sqrt{1+\omega_0^2R^2C^2}} \angle -\beta_1$$

$$\boxed{V_{o1} = \frac{4V_m}{\pi\sqrt{1+\omega_0^2R^2C^2}} \sin(\omega_0 t - \beta_1)}$$

Third harmonic component $(n=3)$

$$\bar{V}_{o3} = \frac{\frac{4V_m}{3\pi}\angle 0^\circ}{1+j3\omega_0 RC} = \frac{4V_m \angle -\beta_3}{3\pi\sqrt{1+9\omega_0^2R^2C^2}}$$

$$\beta_3 = \tan^{-1}(3\omega_0 RC)$$

$$\boxed{V_{o3} = \frac{4V_m}{3\pi\sqrt{1+9\omega_0^2R^2C^2}} \sin(3\omega_0 t - \beta_3)}$$

In General for the k^{th} harmonic component

$$V_{ok} = \frac{4V_m}{k\pi\sqrt{1+k^2\omega_0^2R^2C^2}} \sin(k\omega_0 t - \beta_k) \quad (k \text{ odd})$$

$$\beta_k = \tan^{-1}(k\omega_0 RC) \quad (k \text{ odd})$$

Fourier Series Representation (sum V_{ok}'s)

$$v_o(t) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{4V_m}{n\pi\sqrt{1+n^2\omega_0^2R^2C^2}} \sin(n\omega_0 t - \beta_n)$$

$$\boxed{v_o(t) = \frac{4V_m}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{\sin(n\omega_0 t - \beta_n)}{n\sqrt{1+(n\omega_0 RC)^2}}}$$