CIRCUITS II

Chapter 17: The Fourier Transform

Exponential Form of Fourier series $f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$ where $C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) \times e^{-jn\omega_0 t}\, dt$

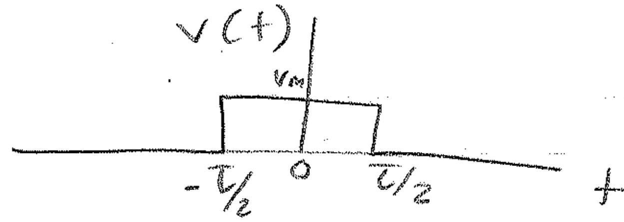
Fourier Transform of $f(t)$

$$\boxed{F(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt}$$

Inverse Fourier Transform

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, dt}$$

For a block pulse



$$F(\omega) = \int_{-\tau/2}^{\tau/2} V_m e^{-j\omega t}\, dt$$

$$= V_m\left(\frac{1}{-j\omega}\right) e^{-j\omega t}\Big|_{-\tau/2}^{\tau/2}$$

$$= \frac{V_m}{-j\omega}\Big(\underbrace{e^{\frac{-j\omega\tau}{2}} - e^{\frac{j\omega\tau}{2}}}_{-2j\sin\left(\frac{\omega\tau}{2}\right)}\Big)$$

$$F(\omega) = \frac{V_m}{-j\omega}\left(-2j\sin\frac{\omega\tau}{2}\right)$$

$$= \frac{2V_m}{\omega}\sin\left(\frac{\omega\tau}{2}\right)$$

Multiply numerator and denominator by τ

$$F(\omega) = \frac{V_m\tau}{\omega\tau/2}\sin\left(\frac{\omega\tau}{2}\right)$$

$$\boxed{F(\omega) = V_m\tau\,\frac{\sin\left(\frac{\omega\tau}{2}\right)}{(\omega\tau/2)}}$$

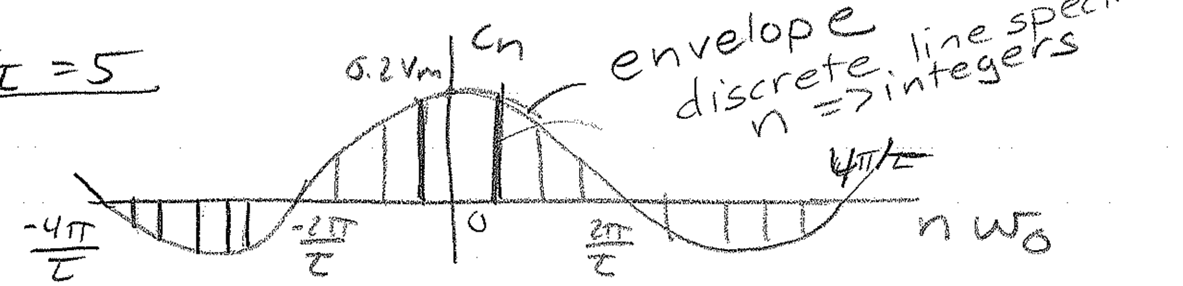
CIRCUITS II

For a Periodic Train of voltage pulses

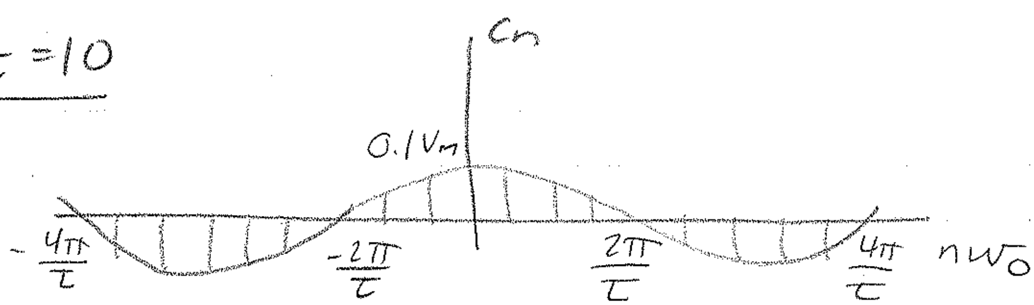
$$C_n = \frac{V_m \tau}{T} \frac{\sin(n\omega_0 \tau/2)}{n\omega_0 \tau/2}$$

Plot C_n vs $n\omega_0$ for ratios of T/τ

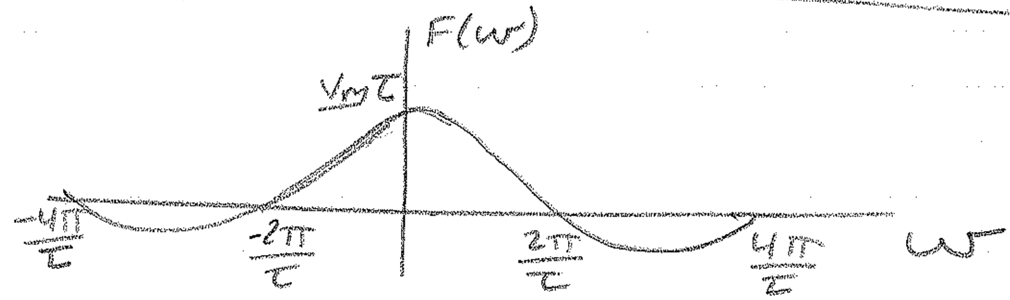
$T/\tau = 5$



$T/\tau = 10$



For $F(\omega)$, Continuous Line Spectrum



Drill Exercise 17.1

Drill Exercise 17.1

Find the Fourier transform of

(a) $f(t) = -A \qquad -\tau/2 \le t \le 0$

$f(t) = A \qquad 0 < t < \tau/2$

$f(t) = 0 \qquad$ elsewhere

$$F(\omega) = \int_{-\tau/2}^{0} -A e^{-j\omega t} dt + \int_{0}^{\tau/2} A e^{-j\omega t} dt$$

$$= \frac{A}{j\omega} e^{-j\omega t}\Big|_{-\tau/2}^{0} + \frac{-A}{j\omega} e^{-j\omega t}\Big|_{0}^{\tau/2}$$

$$= \frac{A}{j\omega}\left(1 - e^{j\omega\tau/2}\right) - \frac{A}{j\omega}\left(e^{-j\omega\tau/2} - 1\right)$$

$$= \frac{A}{j\omega}\left[2 - \underbrace{\left(e^{j\omega\tau/2} + e^{-j\omega\tau/2}\right)}_{2\cos\omega\tau/2}\right]$$

$$F(\omega) = \frac{2A}{j\omega}\left(1 - \cos\left(\frac{\omega\tau}{2}\right)\right)$$

$$\boxed{F(\omega) = -j\frac{2A}{\omega}\left(1 - \cos\left(\frac{\omega\tau}{2}\right)\right)}$$

CIRCUITS II

Convergence of ~~The~~ the Fourier Integral

Fourier transform of a constant

$f(t) = Ae^{-\varepsilon|t|} \quad \varepsilon > 0$

$$F(\omega) = \int_{-\infty}^{0} Ae^{\varepsilon t} e^{-j\omega t}\, dt + \int_{0}^{\infty} Ae^{-\varepsilon t} e^{-j\omega t}\, dt$$

$$= \frac{A}{\varepsilon - j\omega} + \frac{A}{\varepsilon + j\omega} = \frac{2\varepsilon A}{\varepsilon^2 + \omega}$$

@ $\omega = 0$, as $\varepsilon \to 0$ => $F(\omega)$ appoaches the impulse function, $\delta(\omega)$

Strength of the impulse => Area under $F(\omega)$

$$\int_{-\infty}^{\infty} \frac{2\varepsilon A}{\varepsilon^2 + \omega^2}\, d\omega = 4\varepsilon A \int_{0}^{\infty} \frac{d\omega}{\varepsilon^2 + \omega^2}$$

$$= 4\varepsilon A \left(\frac{1}{\varepsilon}\right) \tan^{-1}\left(\frac{\omega}{a}\right)\Bigg|_0^{\infty}$$

$$= 4A\left(\frac{\pi}{2} - 0\right) = 2\pi A$$

$$\boxed{\mathcal{F}\{A\} = 2\pi A\, \delta(\omega)}$$

72

CIRCUITS II

Using Laplace Transforms to find the Fourier Transform

$$\mathcal{F}\{f(t)\} = \mathcal{L}\{f(t)\}\Big|_{s=j\omega}$$

$f(t) = 0 \qquad t \le 0^-$

$f(t) = e^{-at}\cos\omega_0 t \qquad t \ge 0^+$

$$\mathcal{F}\{f(t)\} = \frac{s+a}{(s+a)^2+\omega_0^2}\Bigg|_{s=j\omega}$$

$$\boxed{\mathcal{F}\{f(t)\} = \frac{j\omega + a}{(j\omega+a)^2+\omega_0^2}}$$

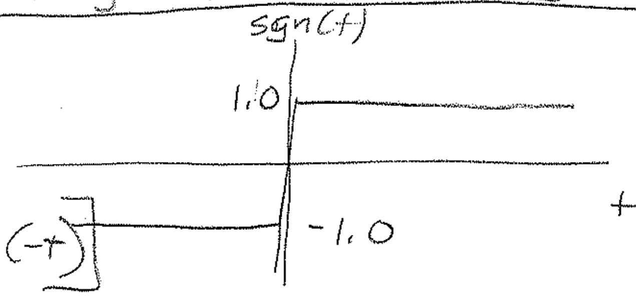
Negative time $\Rightarrow$ use $s = -j\omega$

$f^+(t) = f(t) \quad t > 0 \qquad f(t) = f^+(t) + f^-(t)$

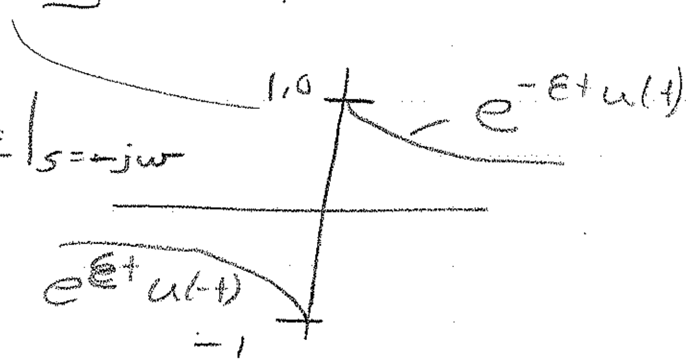
$f^-(t) = f(t) \quad t < 0 \qquad \mathcal{F}(f(t)) = \mathcal{L}\{f^+(t)\}|_{s=j\omega} + \mathcal{L}\{f^-(-t)\}|_{s=-j\omega}$

Fourier Transfor of a Signum Function sgn(t)

$sgn(t) = u(t) - u(-t)$



$$sgn(t) = \lim_{\varepsilon \to 0}\left[e^{-\varepsilon t}u(t) - e^{\varepsilon t}u(-t)\right]$$



$$\mathcal{F}\{f(t)\} = \frac{1}{s+\varepsilon}\Big|_{s=j\omega} - \frac{1}{s+\varepsilon}\Big|_{s=-j\omega}$$

$$= \frac{1}{j\omega+\varepsilon} - \frac{1}{-j\omega+\varepsilon}$$

$$= \frac{-2j\omega}{\omega^2+\varepsilon^2}$$

As $\varepsilon \to 0 \Rightarrow \mathcal{F}\{f(t)\} \to \frac{2}{j\omega} = \boxed{\mathcal{F}\{sgn(t)\} = \frac{2}{j\omega}}$

1

Drill Exercise 17.3

(c) $f(t) = te^{-at} \quad f(t) \geq 0$
$f(t) \quad te^{at} \quad f(t) \leq 0$

$$\mathcal{F}\{f(t)\} = \mathcal{L}\{te^{-at}\}\Big|_{s=j\omega} + \mathcal{L}\{-te^{-at}\}\Big|_{s=-j\omega}$$

$$= \frac{1}{(s+a)^2}\Big|_{s=j\omega} + \frac{-1}{(s+a^2)}\Big|_{s=-j\omega}$$

$$F(\omega) = \frac{1}{(a+j\omega)^2} - \frac{1}{(a-j\omega)^2} = \frac{(a-j\omega)^2 - (a+j\omega)^2}{(a+j\omega)^2\,(a-j\omega)^2}$$

$$\boxed{F(\omega) = \frac{-j4a\omega}{(a^2+\omega^2)}}$$

(b) $f(t) = 0 \quad t > 0$
$f(t) = -te^{at} \quad t \leq 0$

$$F(\omega) = \mathcal{L}\{+te^{-at}\}\Big|_{s=-j\omega}$$

$$= \frac{1}{(s+a)^2}\Big|_{s=-j\omega}$$

$$\boxed{F(\omega) = \frac{1}{(a-j\omega)^2}}$$

(a) $f(t) = 0 \quad t < 0$
$f(t) = e^{-at}\sin\omega_0 t \quad t \geq 0$

$$F(\omega) = \mathcal{L}\{e^{-at}\sin\omega_0 t\}\Big|_{s=j\omega} = \frac{\omega_0}{(s+a)^2+\omega_0^2}\Big|_{s=j\omega}$$

$$\boxed{F(\omega) = \frac{\omega_0}{(a+j\omega)^2+\omega_0^2}}$$

Unit Step Function

$$u(t) = \frac{1}{2} + \frac{1}{2}\,\mathrm{sgn}(t)$$

$$\mathcal{F}\{u(t)\} = \mathcal{F}\{\tfrac{1}{2}\} + \mathcal{F}\{\tfrac{1}{2}\,\mathrm{sgn}(t)\}$$

$$\boxed{\mathcal{F}\{u(t)\} = \pi\,\delta(\omega) + \frac{1}{j\omega}}$$

Cosine Function

$$\mathcal{F}\{A\} = 2\pi A\,\delta(\omega)$$

$$\mathcal{F}\{e^{j\omega_0 t}\} = 2\pi\,\delta(\omega-\omega_0)$$

$$F(\omega) = \int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} A e^{-\varepsilon|t|} e^{-j(\omega-\omega_0)t}\,dt$$

$A = 1$

$$\cos\omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$$\mathcal{F}\{\cos\omega_0 t\} = \frac{1}{2}\left(\mathcal{F}\{e^{j\omega_0 t}\} + \mathcal{F}\{e^{-j\omega_0 t}\}\right)$$

$$\boxed{\mathcal{F}\{\cos\omega_0 t\} = \pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega-\omega_0)}$$

Fourier Transforms => Table 17.1

Operational Transforms

Multiplication by a Constant

$$\mathcal{F}\{K f(t)\} = K\,F(\omega)$$

Addition / Subtraction

$$\mathcal{F}\{f_1(t)\} = F_1(\omega), \quad \mathcal{F}\{f_2(t)\} = F_2(\omega)$$

$$\mathcal{F}\{f_1(t) - f_2(t)\} = F_1(\omega) - F_2(\omega)$$

CIRCUITS II

Differentiation

$$\mathcal{F}\left\{\frac{df(t)}{dt}\right\} = j\omega F(\omega)$$

$$\mathcal{F}\left\{\frac{d^n f(t)}{dt^n}\right\} = (j\omega)^n F(\omega)$$

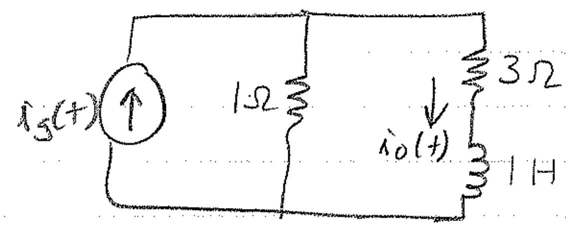
Other Operational Transforms => Table 17.2

Circuit Applications

=> Laplace Tranform more commonly used.

Output $Y(\omega) = X(\omega)\, H(\omega)$ $s = j\omega$

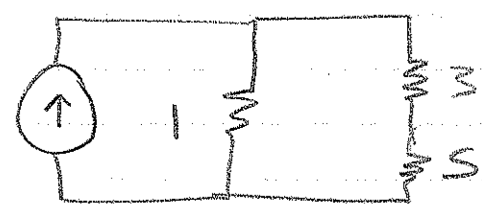
Example 17.1 input transfer function



$i_g(t) = 20\,\mathrm{sgn}(t)$ A

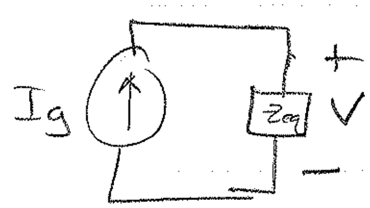
$$I_g(\omega) = 20\left(\frac{2}{j\omega}\right) = \frac{40}{j\omega}$$

$H(\omega) =$ for I_o to I_g



$$\frac{1}{Z_{eq}} = \left(1 + \frac{1}{s+3}\right)^{-1} = \frac{s+3+1}{s+3}$$

$$Z_{eq} = \left(\frac{s+4}{s+3}\right)^{-1} = \frac{s+3}{s+4}$$



$$V = Z_{eq} I_g = \frac{s+3}{s+4} I_g$$

$$I_o = \frac{V}{s+3} = \frac{I_g}{s+4}$$

$$H(s) = \frac{I_o}{I_g} = \frac{1}{s+4} \qquad H(\omega) = H(s)\Big|_{s=j\omega} = \frac{1}{4+j\omega}$$

75

CIRCUITS II

$$I_o(\omega) = I_g(\omega)\,H(\omega)$$

$$= \left(\frac{40}{j\omega}\right)\left(\frac{1}{4+j\omega}\right)$$

$$\boxed{I_o(\omega) = \frac{40}{j\omega(4+j\omega)}}$$

$$I_o(\omega) = \frac{K_1}{j\omega} + \frac{K_2}{4+j\omega}$$

$$K_1 = 10,\ K_2 = -10$$

$$I_o(\omega) = \frac{10}{j\omega} - \frac{10}{4+j\omega}$$

$$= 5\left(\frac{2}{j\omega}\right) - 10\left(\frac{1}{4+j\omega}\right)$$

$$\boxed{i_o(t) = 5\,\mathrm{sgn}(t) - 10e^{-4t}u(t)}$$