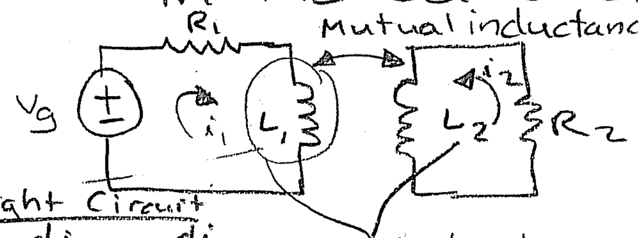
①

CIRCUITS II

Mutual Inductance (6.4)

- Two circuits are linked by a magnetic field
- Induced voltage in one circuit is related by the current in the other circuit through mutual inductance

- Self-Inductance - relates voltage to current in the same circuit



Left Circuit
Induced voltages $L_1 \frac{di_1}{dt}$, $M\frac{di_2}{dt}$

Right Circuit
$L_2\frac{di_2}{dt}$, $M\frac{di_1}{dt}$

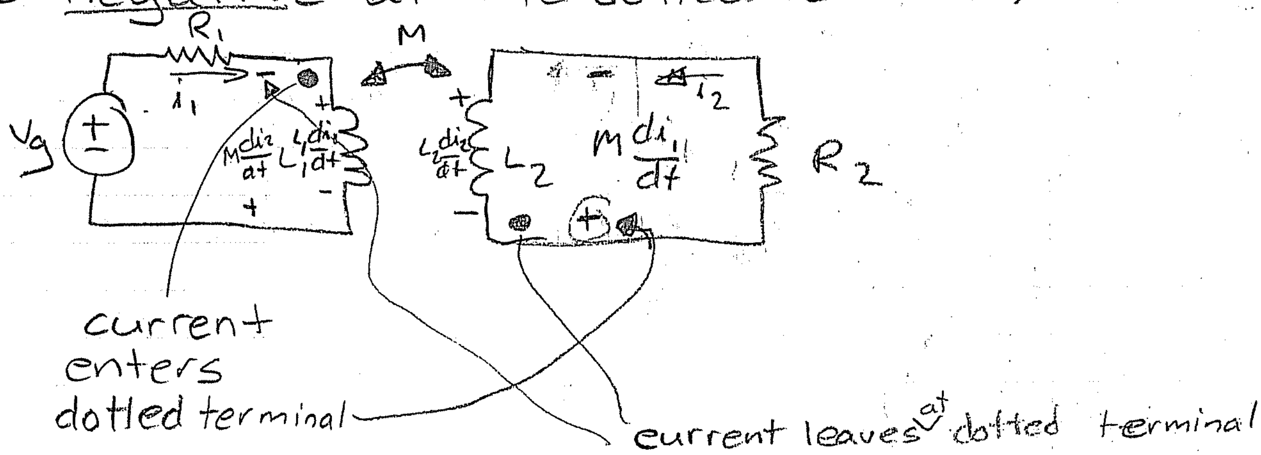
Dot Convention for Mutual Inductance

- Provides information on polarities

Rules

- When the reference direction for the current enters the dotted terminal of the coil, the polarity of the voltage of the other coil is positive at the dotted terminal

- When the reference direction for the current leaves the dotted terminal of the coil, the polarity of the voltage of the other coil is negative at the dotted terminal.



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CIRCUITS II

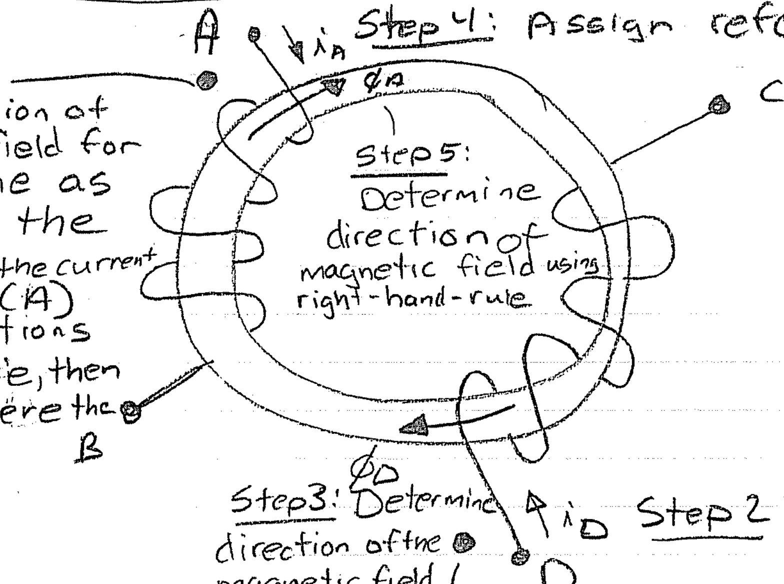
Mesh Current Method

$$-v_g + i_1 R_1 + L_1 \frac{di_1}{dt} - M\frac{di_2}{dt} = 0 \qquad \text{(Left Mesh)}$$

$$i_2 R_2 + L_2 \frac{di_2}{dt} - M\frac{di_1}{dt} = 0 \qquad \text{(Right Mesh)}$$

Determining Dot Markings

Step 4: Assign refencence current direction



Step 6
- If the direction of the magnetic field for A is the same as for D, then the dot is where the current enters the coil (A)
- If the directions are opposite, then the dot is where the current leaves the coil

Step 3: Determine direction of the magnetic field using right-hand rule

Step 2: Assign reference current direction

Step 1: Pick a terminal to be dotted (Arbitrary)

Self Inductance

Faraday's Law

$$V = \frac{d\lambda}{dt}$$

λ = flux linkage measured in weber-turns

$$\lambda = N\phi$$

N = number of turns linked by the field

ϕ = magnetic flax

$$\phi = \mathcal{P} N i$$

$\mathcal{P}$ = permeanee of the space occupied by flux (magnetic properties)

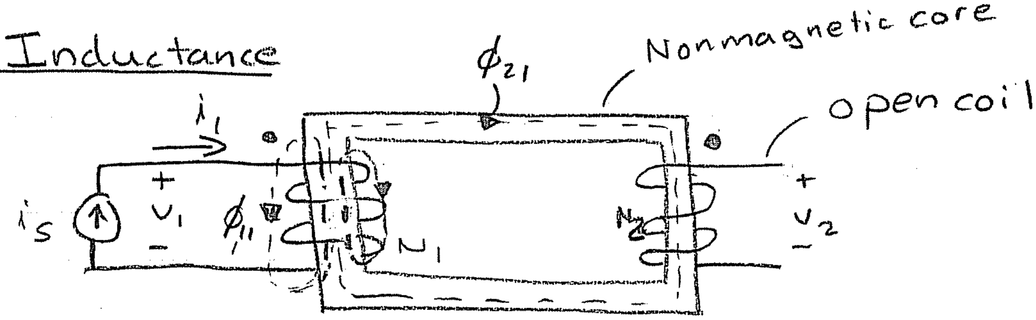
CIRCUITS II

N = number of turns in the coil

$$v = \frac{d\lambda}{dt} = \frac{d(N\phi)}{dt} = N\frac{d\phi}{dt} = N\frac{d}{dt}(\mathcal{P}Ni)$$

$$v = \underbrace{N^2\mathcal{P}}_{L}\frac{di}{dt} \Rightarrow v = L\frac{di}{dt}$$

Mutual Inductance



Total flux in coil 1, ϕ_1

$$\phi_1 = \phi_{11} + \phi_{21}$$

ϕ_{11} = produced by i_1 in coil 1 that links N_1

ϕ_{21} = produced by i_1 in coil 1, but links N_2

$$\phi_1 = \mathcal{P}_1 N_1 i_1$$

$$\phi_{11} = \mathcal{P}_{11} N_1 i_1$$

$$\phi_{21} = \mathcal{P}_{21} N_1 i_1$$

$$\mathcal{P}_1 = \mathcal{P}_{11} + \mathcal{P}_{21}$$

$$v_1 = \frac{d\lambda_1}{dt} = \frac{d(N_1\phi_1)}{dt} = N_1\frac{d}{dt}(\phi_{11} + \phi_{21})$$

$$= N_1\frac{d}{dt}(N_1\mathcal{P}_{11}i_1 + N_1\mathcal{P}_{21}i_1) = \underbrace{N_1^2\mathcal{P}_1}_{L_1}\frac{di_1}{dt} = \boxed{L_1\frac{di_1}{dt}}$$

$$v_2 = \frac{d\lambda_2}{dt} = \frac{d(N_2\phi_{21})}{dt} = N_2\frac{d}{dt}(N_1\mathcal{P}_{21}i_1)$$

$$v_2 = \underbrace{N_2N_1\mathcal{P}_{21}}_{M_{21}}\frac{di_1}{dt} \qquad M_{21} = N_2N_1\mathcal{P}_{21}$$

Similarly

$$M_{12} = N_1N_2\mathcal{P}_{12}$$

For nonmagnetic cores

$$\mathcal{P}_{12} = \mathcal{P}_{21} \Rightarrow \boxed{M_{12} = M_{21}}$$

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CIRCUITS II

$$L_1 = N_1^2 \mathcal{P}_1 = N_1^2(\mathcal{P}_{11} + \mathcal{P}_{21})$$

$$L_2 = N_2^2 \mathcal{P}_2 = N_2^2(\mathcal{P}_{22} + \mathcal{P}_{12})$$

$$L_1 L_2 = N_1^2 N_2^2 (\mathcal{P}_{11} + \mathcal{P}_{21})(\mathcal{P}_{22} + \mathcal{P}_{12})$$

$$= M^2 \underbrace{\left(1 + \frac{\mathcal{P}_{11}}{\mathcal{P}_{12}}\right)\left(1 + \frac{\mathcal{P}_{22}}{\mathcal{P}_{12}}\right)}_{1/K^2}$$

$K \Rightarrow$ coefficient of coupling

$$M^2 = K^2 L_1 L_2$$

$$\boxed{M = K\sqrt{L_1 L_2}} \qquad 0 \le K \le 1$$

Energy

$$w(t) = \tfrac{1}{2} L_1 i_1^2 + \tfrac{1}{2} L_2 i_2^2 \pm M i_1 i_2$$

If both currents enter the polarity marked terminals, $\Rightarrow + M i_1 i_2$

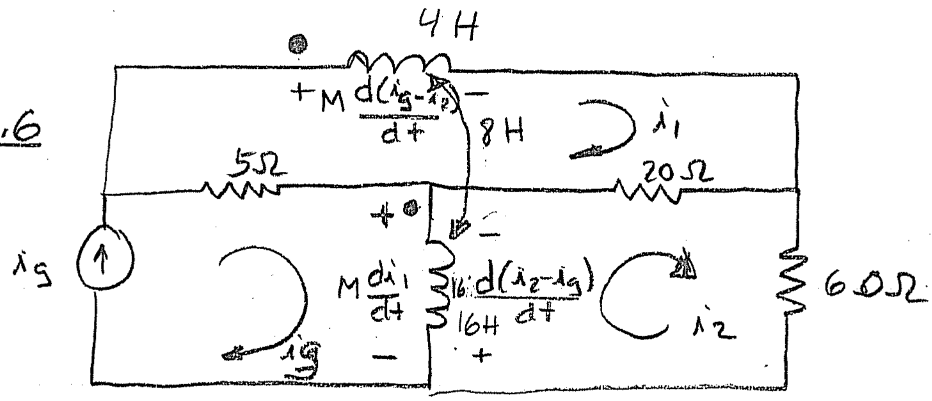
If one enters while the other leaves, $\Rightarrow - M i_1 i_2$

If both leaving $\Rightarrow + M i_1 i_2$

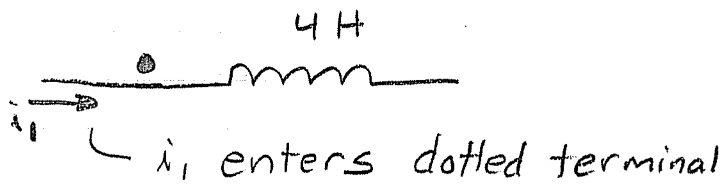
Example 6.6
Drill Exercise 6.8

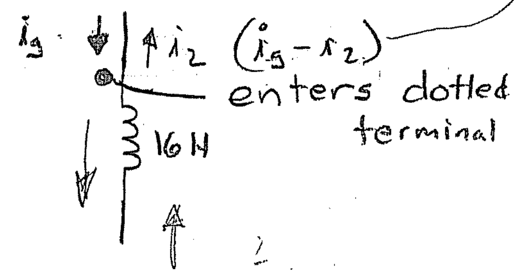
①

Example 6.6



(a) Write a set of Mesh-Current Equations that describe the circuit in terms of i_1 & i_2





Mesh 1

$$(4H)\frac{di_1}{dt} + (8H)\frac{d(i_g - i_2)}{dt} + (20\Omega)(i_1 - i_2) + (5\Omega)(i_1 - i_g) = 0$$

Mesh 2

$$(20\Omega)(i_2 - i_1) + (60\Omega)(i_2) + (16H)\frac{d(i_2 - i_g)}{dt} - (8H)\frac{di_1}{dt} = 0$$

(b) Verify that if there is no energy stored in the circuit at $t=0$ and if $i_g = 16 - 16e^{-5t}$ A, that the solutions for i_1 & i_2 are

$$i_1 = 4 + 64e^{-5t} - 68e^{-4t} \text{ A}$$

$$i_2 = 1 - 52e^{-5t} + 51e^{-4t} \text{ A}$$

No energy at $t=0 \Rightarrow i = 0$

$$i_1(0) = (4 + 64 - 68)\text{A} = 0$$

$$i_2(0) = (1 - 52 + 51)\text{A} = 0 \quad ✓$$

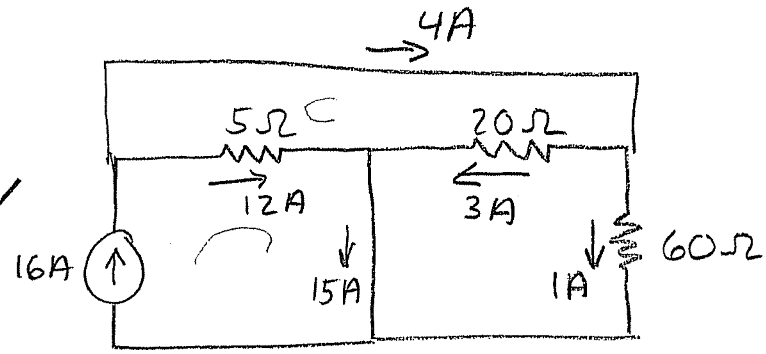
@ $t = \infty$ $\Rightarrow$ voltage in the inductor $= 0$

$i_g = 16$A, $i_1 = 4$A, $i_2 = 1$ A

(2)

Example 6.6

Use KVL + KCL ✓



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Drill Exercise 6.8

Two coupled coils wound around a nonmagnetic core

$L_1 = 288\,mH$, $M = 90\,mH$

$K = 0.75$ $\quad \mathcal{P}_{11} = \mathcal{P}_{22}$

(a) Find L_2 and N_1/N_2 (turns ratio)

$$M = K\sqrt{L_1 L_2}$$

$$L_2 = \frac{1}{L_1}\left(\frac{M}{K}\right)^2 = \frac{1}{288\times10^{-3}}\left(\frac{90\times10^{-3}}{0.75}\right)^2$$

$$L_2 = 0.05\,H = \boxed{50\,mH}$$

$\mathcal{P}_{12} = \mathcal{P}_{21}$ = non magnetic

$\mathcal{P}_{11} = \mathcal{P}_{22}$ (Given)

$$L_1 = N_1^2 \mathcal{P}_1$$

$$L_2 = N_2^2 \mathcal{P}_2$$

$$\frac{L_1}{L_2} = \left(\frac{N_1^2}{N_2^2}\right)\frac{\mathcal{P}_1}{\mathcal{P}_2} = \left(\frac{N_1^2}{N_2^2}\right)\left(\frac{\mathcal{P}_{11} + \mathcal{P}_{12}}{\mathcal{P}_{22} + \mathcal{P}_{21}}\right)$$

$$\frac{N_1}{N_2} = \sqrt{\frac{L_1}{L_2}} = \sqrt{\frac{288\,mH}{50\,mH}} = \boxed{2.4}$$

(b) if $N_1 = 1200$, what is $\mathcal{P}_1$ & $\mathcal{P}_2$

$$\mathcal{P}_1 = \frac{L_1}{N_1^2} = \frac{288\times10^{-3}\,H}{(1200)^2} = \boxed{2\times10^{-7}\ Wb/A}$$

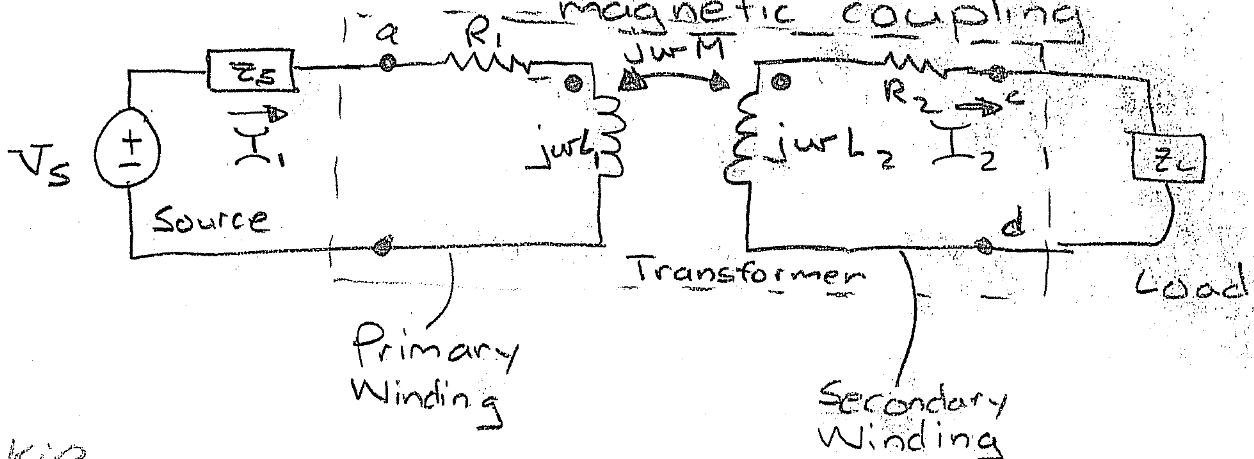
$$N_2 = \frac{N_1}{2.4} = \frac{1200}{2.4} = 500$$

$$\mathcal{P}_2 = \frac{L_2}{N_2^2} = \frac{50\times10^{-3}\,H}{(500)^2} = \boxed{2\times10^{-7}\ Wb/A}$$

CIRCUITS II

Transformers (Sinusoidal Steady-State)

Simple Transformer – Two coils wound around a single core to ensure magnetic coupling



Skip

Mesh-Current

$$V_S = I_1(Z_S + R_1 + j\omega L_1) - I_2(j\omega M)$$

$$I_2(R_2 + j\omega L_2 + Z_L) - j\omega M I_1 = 0$$

Let $Z_{11} = Z_S + R_1 + j\omega L_1$

$Z_{22} = R_2 + j\omega L_2 + Z_L$

A SA

$$I_1 = \frac{Z_{22}}{Z_{11} Z_{22} + \omega^2 M^2} V_S$$

$$I_2 = \frac{j\omega M}{Z_{11} Z_{22} + \omega^2 M^2} V_S = \frac{j\omega M}{Z_{22}} I_1$$

* **Impedance of the Voltage Source, Z_{int}**

$$Z_{int} = V_S / I_1 = \frac{Z_{11} Z_{22} + \omega^2 M^2}{Z_{22}} = \boxed{Z_{11} + \frac{\omega^2 M^2}{Z_{22}} = Z_{int}}$$

Impedance at the source terminals a + b

* $$Z_{ab} = Z_{int} - Z_S = Z_{11} + \frac{\omega^2 M^2}{Z_{22}} - Z_S = Z_S + R_1 + j\omega L_1 + \frac{\omega^2 M^2}{R_2 + j\omega L_2 + Z_L} - Z_S$$

$$\boxed{Z_{ab} = j\omega L_1 + \frac{\omega^2 M^2}{R_2 + j\omega L_2 + Z_L}}$$

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CIRCUITS II

* Reflected Impedance, Z_r

Defined as $\dfrac{\omega^2 M^2}{R_2 + j\omega L_2 + Z_L}$

Let the load impedance, $Z_L = R_L + jX_L$

$$Z_r = \frac{\omega^2 M^2}{R_2 + R_L + j(\omega L_2 + X_L)} \quad \text{— conjugate}$$

$$Z_r = \frac{\omega^2 M^2}{|Z_{22}|^2}\left[(R_2+R_L) - j(\omega L_2 + X_L)\right]$$

Note: $Z_{22} = R_2 + R_L + j(\omega L_2 + X_L)$

$$\therefore \quad |Z_{22}|^2 = \left[(R_2+R_L) + j(\omega L_2 + X_L)\right]\left[(R_2+R_L) - j(\omega L_2 + X_L)\right]$$

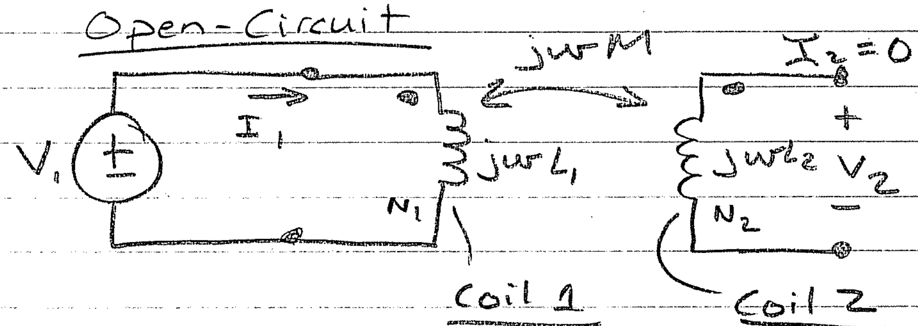
Start

Ideal Transformer

- $K = 1$
- $L_1 = L_2 = \infty$ (magnetic coupling)
- Coil Losses are negligible

Voltage and Current Ratios

Open-Circuit



Coil 2, $V_2 = j\omega M I_1$

Coil 1 $\quad I_1 = \dfrac{V_1}{j\omega L_1}$

Substituting

$$V_2 = \frac{M}{L_1} V_1$$

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CIRCUITS II

Ideal Transformer

$k = 1 \qquad M = \sqrt{L_1 L_2}$

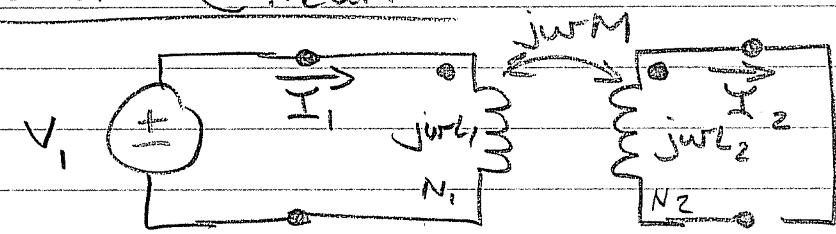
$$V_2 = \frac{\sqrt{L_1 L_2}}{L_1} V_1 \Rightarrow \boxed{V_2 = \sqrt{\frac{L_2}{L_1}}\, V_1}$$

$L_1 = N_1^2 \mathcal{P}$

$$V_2 = \sqrt{\frac{N_2^2 \mathcal{P}}{N_1^2 \mathcal{P}}}\, V_1 \Rightarrow \boxed{V_2 = \frac{N_2}{N_1} V_1}$$

$$\boxed{\frac{V_1}{N_1} = \frac{V_2}{N_2}}$$

Short Circuit



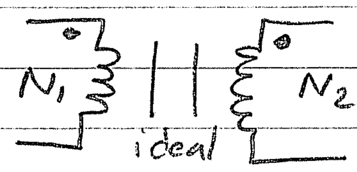
KVL (Coil 2)

$$0 = -j\omega M I_1 + j\omega L_2 I_2$$

$$\frac{I_1}{I_2} = \frac{L_2}{M} = \frac{L_2}{\sqrt{L_1 L_2}} = \sqrt{\frac{L_2}{L_1}} = \frac{N_2}{N_1}$$

$$\boxed{I_1 N_1 = I_2 N_2}$$

Symbol for and Ideal Transformer



(8)

CIRCUITS II

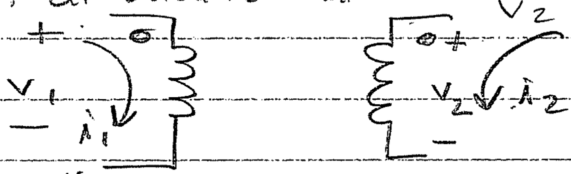
Polarity of the Voltage and Current Ratios

Rules

1) If v_1 & v_2 are both positive or negative at the dotted terminal, use a plus (+) sign $\frac{v_1}{N_1} = \frac{v_2}{N_2}$. If not, use a negative sign
2) If i_1 & i_2 are both directed in or out of the dotted terminals, use a minus sign. If not, use a plus sign

$i_1 \Rightarrow$ into dotted terminal $\qquad$ $i_2 \Rightarrow$ into dotted terminal

$v_1 \Rightarrow +$ at dotted terminal $\qquad$ $v_2 \Rightarrow +$ at dotted terminal



$$\frac{v_1}{N_1} = \frac{v_2}{N_2}$$

$$N_1 i_1 = -N_2 i_2$$

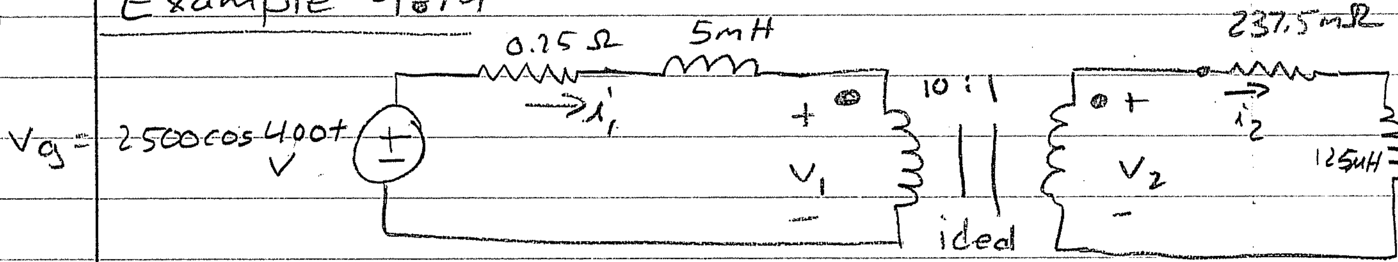
Turns ratio

$$\boxed{a = \frac{N_2}{N_1}}$$

where a = turns ratio

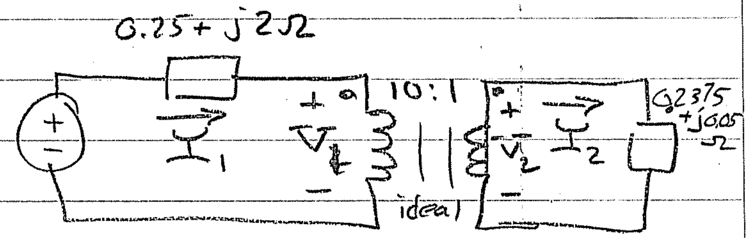
1

Example 9.14



Find: i_1, v_1, v_2, v_2

$$\mathbf{V}_g = 2500\angle 0^\circ \text{ V}$$



Ideal Transformer

$$\frac{V_1}{N_1} = \frac{V_2}{N_2} \qquad \mathbf{V}_1 = \frac{N_1}{N_2}\mathbf{V}_2$$

$$\boxed{\mathbf{V}_1 = 10\mathbf{V}_2}$$

$$N_1 i_1 = N_2 i_2 \qquad \mathbf{I}_2 = \frac{N_1}{N_2}\mathbf{I}_1 \Rightarrow \boxed{\mathbf{I}_2 = 10\mathbf{I}_1}$$

KVL Left Circuit

$$-2500\angle 0^\circ + (0.25 + j2)\mathbf{I}_1 + \mathbf{V}_1 = 0 \quad (1)$$

Right Circuit

$$-\mathbf{V}_2 + (0.2375 + j0.05)\mathbf{I}_2 = 0$$

$$\mathbf{V}_2 = (0.2375 + j0.05)\mathbf{I}_2 = (0.2375 + j0.05)(10\mathbf{I}_1)$$

$$\left(\frac{\mathbf{V}_1}{10}\right) = 10\mathbf{I}_1(0.2375 + j0.05)$$

$$\mathbf{V}_1 = 100\mathbf{I}_1(0.2375 + j0.05)$$

②

Example 9.14

Substitute into ①

$$2500\angle 0^\circ = (0.25 + j2)\mathbf{I}_1 + (23.75 + j5)\mathbf{I}_1$$

$$2500\angle 0^\circ = (24 + j7)\mathbf{I}_1$$

$Z_m \cos\phi = 24$

$Z_m \sin\phi = 7$

$\tan\phi = \frac{7}{24}$

$\phi = 16.26^\circ$

$Z_m = \frac{24}{\cos 16.26^\circ} = 25$

$$2500\angle 0^\circ = 25\angle 16.26^\circ \ \mathbf{I}_1$$

$$\mathbf{I}_1 = \frac{2500\angle 0^\circ}{25\angle 16.26}$$

$$\boxed{\mathbf{I}_1 = 100\angle -16.26^\circ \text{ A}}$$

$$\boxed{i_1 = 100\cos(400t - 16.26^\circ) \text{ A}}$$

$$\mathbf{I}_2 = 10\,\mathbf{I}_1 = 1000\angle -16.26^\circ \text{ A}$$

$$\boxed{i_2 = 1000\cos(400t - 16.26^\circ) \text{ A}}$$

$$\mathbf{V}_1 = 100(0.2375 + j0.05)\mathbf{I}_1$$

$$\mathbf{V}_1 = (23.75 + j5)(100\angle -16.26^\circ)$$

$$= (24.27\angle 11.89^\circ)(100\angle -16.26^\circ)$$

$$\boxed{\mathbf{V}_1 = 2427\angle -4.37^\circ \text{ V}} \Rightarrow \boxed{v_1 = 2427\cos(400t - 4.37^\circ) \text{ V}}$$

$$\mathbf{V}_2 = \frac{1}{10}\mathbf{V}_1 = 242.7\angle -4.37^\circ$$

$$\boxed{v_2 = 242.7\cos(400t - 4.37^\circ) \text{ V}}$$