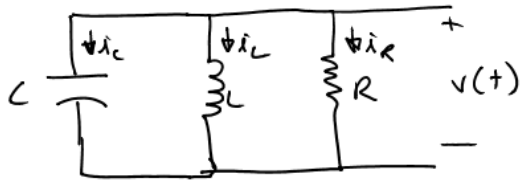
Natural Response of a Parallel RLC Circuit



KCL

$$i_C + i_L + i_R = 0$$

$$C\frac{dv}{dt} + \left(\frac{1}{L}\int_0^t v\,dt + I_0\right) + \frac{v}{R} = 0$$

Differentiate with respect to time

$$C\frac{d^2v}{dt^2} + \frac{1}{L}v + \frac{1}{R}\frac{dv}{dt} = 0$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

Second Order Circuit

Constants

Guess $\quad v(t) = Ae^{st}$

$$\frac{dv}{dt} = Ase^{st}$$

$$\frac{d^2v}{dt^2} = As^2e^{st}$$

$$As^2e^{st} + \frac{1}{RC}\left(Ase^{st}\right) + \frac{1}{LC}\left(Ae^{st}\right) = 0$$

$$\underbrace{Ae^{st}}_{=0}\underbrace{\left(s^2 + \frac{1}{RC}s + \frac{1}{LC}\right)}_{=0} = 0$$

$\longrightarrow A = 0 \Rightarrow v = 0 \Rightarrow$ Trivial Solution

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \quad \text{(Characteristic Equation)}$$

Quadratic Formula

$$s = \frac{-\frac{1}{RC} \pm \sqrt{\left(\frac{1}{RC}\right)^2 - (4)\left(\frac{1}{LC}\right)}}{2}$$

$$s = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$\alpha = \frac{1}{2RC}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

Case 1: Overdamped

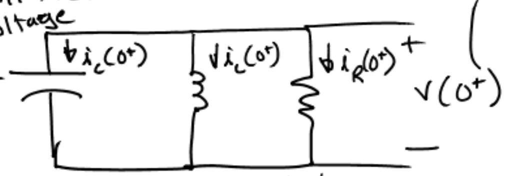
$\omega_0^2 < \alpha^2$ => Two real, unique roots
=> Overdamped

$$\boxed{v = A_1 e^{s_1 t} + A_2 e^{s_2 t}}$$

A_1 & A_2 => Initial Conditions

just after $t=0$

$\boxed{v(0^+) = V_0}$ – Initial voltage in the capacitor



$i_L(0^+) = I_0$ – Initial current in the inductor

$$i_R(0^+) = \frac{V_0}{R}$$

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$$i_C(0^+) + i_L(0^+) + i_R(0^+) = 0$$

$$C\frac{dv(0^+)}{dt} + I_0 + \frac{V_0}{R} = 0$$

$$\boxed{\frac{dv(0^+)}{dt} = \frac{1}{C}\left(-I_0 - \frac{V_0}{R}\right)}$$

Case 2: Underdamped

$\omega_0^2 > \alpha^2 \quad \Rightarrow \sqrt{\text{Negative Number}}$

$\Rightarrow$ Underdamped

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$= -\alpha + \sqrt{(-1)(\omega_0^2 - \alpha^2)}$$

$$= -\alpha + \sqrt{-1}\,\underbrace{\sqrt{\omega_0^2 - \alpha^2}}_{\omega_d}$$

$\sqrt{-1} = j$ (i in math)

$$s_1 = -\alpha + j\omega_d \qquad s_2 = -\alpha - j\omega_d$$

$$v(t) = Ae^{st}$$

$$v(t) = B_1 e^{(-\alpha + j\omega_d)t} + B_2 e^{(-\alpha - j\omega_d)t}$$

Using $e^{\pm j\theta} = \cos\theta + j\sin\theta$

$$\boxed{v(t) = B_1 e^{-\alpha t}\cos(\omega_d t) + B_2 e^{-\alpha t}\sin(\omega_d t)}$$

B_1 & B_2 $\Rightarrow$ Initial Conditions

Case 3: Critically Damped

$\omega_0^2 = \alpha^2 \Rightarrow$ Double Root

$\Rightarrow$ Critically Damped

$s_1 = s_2 = -\alpha$

$$\boxed{v(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}}$$

D_1 & D_2 => Initial Conditions

Note on Determining the Current in the Inductor

$i_R(t),\ i_C(t),\ i_L(t)$

$$i_R(t) = \frac{v(t)}{R}$$

$$i_C(t) = C\frac{dv}{dt}$$

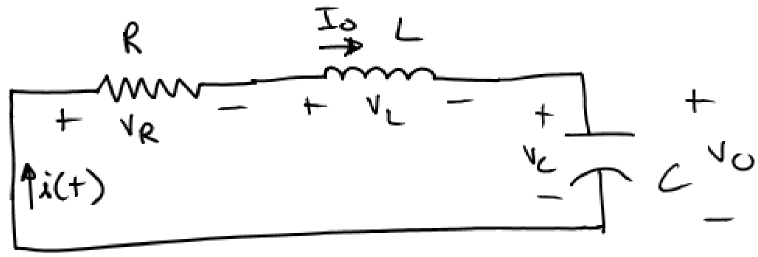
$$\cancel{i_L(t) = \frac{1}{L}\int_0^t v\,dt + I_0}$$

KCL

$$i_C(t) + i_R(t) + i_L(t) = 0$$

$$\boxed{i_L(t) = -i_C(t) - i_R(t)}$$

Natural Response of a Series RLC Circuit



$\underline{KVL}$

$$v_R + v_L + v_C = 0$$

$$iR + L\frac{di}{dt} + \left(\frac{1}{C}\int_0^t i\,dt + V_0\right) = 0$$

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC} i = 0$$

$$s = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$\alpha = \frac{R}{2L} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

<u>Case 1: Overdamped</u>

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

<u>Case 2: Underdamped</u>

$$i(t) = B_1 e^{-\alpha t}\cos\omega_d t + B_2 e^{-\alpha t}\sin\omega_d t$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

<u>Case 3: Critically Damped</u>

$$i(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

Solving for the Constants Using Initial Conditions

1. $i(0^+) = I_0$

2. KVL

$$v_R(0^+) + v_L(0^+) + v_C(0^+) = 0$$

$$I_0 R + L\frac{di(0^+)}{dt} + V_0 = 0$$

$$\boxed{\frac{di(0^+)}{dt} = \frac{1}{L}\left(-I_0 R - V_0\right)}$$

Note on Determining the Voltage in the Capacitor

$$v_R(t) = iR$$

$$v_L(t) = L\frac{di}{dt}$$

$$\cancel{v_C(t) = \frac{1}{C}\int_0^t i\,dt + V_0}$$

KVL

$$v_R(t) + v_L(t) + v_C(t) = 0$$

$$\boxed{v_C(t) = -v_R(t) - v_L(t)}$$