

2.3 Calculating Limits Using the Limit Laws

In the previous section, we saw that just using graphs and your calculator will not always be accurate to find limits.

LIMIT LAWS

Suppose c is a constant and that $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ both exist.

1. $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$ “Limit of a sum is the sum of the limits”
2. $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$ “Limit of a difference is the difference of the limits”
3. $\lim_{x \rightarrow a} c \cdot f(x) = c \cdot \lim_{x \rightarrow a} f(x)$ “Limit of a constant times a function is the constant times the limit”
4. $\lim_{x \rightarrow a} f(x) \cdot g(x) = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$ “Limit of a product is the product of the limits”
5. $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$, provided $\lim_{x \rightarrow a} g(x) \neq 0$ “Limit of a quotient is the quotient of the limits”
6. $\lim_{x \rightarrow a} [f(x)]^n = \left[\lim_{x \rightarrow a} f(x) \right]^n$ (we use law #4 to derive this one)
7. $\lim_{x \rightarrow a} c = c$ (consider the graph $y = c$)
8. $\lim_{x \rightarrow a} x = a$ (consider the graph $y = x$)
9. $\lim_{x \rightarrow a} x^n = a^n$ (use laws #6 and #8)
10. $\lim_{x \rightarrow a} \sqrt[n]{x} = \sqrt[n]{a}$
11. $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$ where n is pos integer, and if n is even then we assume $\lim_{x \rightarrow a} f(x) > 0$.

Ex) Use the limit laws to evaluate $\lim_{x \rightarrow 4} 2x^3 - 7x + 1 = \lim_{x \rightarrow 4} 2x^3 - \lim_{x \rightarrow 4} 7x + \lim_{x \rightarrow 4} 1$

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$$= 2 \lim_{x \rightarrow 4} x^3 - 7 \lim_{x \rightarrow 4} x + 1$$

$$= 2 (\lim_{x \rightarrow 4} x)^3 - 7(4) + 1$$

$$= 2 \cdot 4^3 - 28 + 1$$

$$= 128 - 28 + 1 = 101$$

Now, let $h(x) = 2x^3 - 7x + 1$. Find $h(4)$. What do you notice?

$$h(4) = 101$$

This leads to a handy result for polynomial and rational functions.

***Direct Substitution Property

If f is a polynomial or rational function & $a \in \text{domain}(f)$

$$\text{then } \lim_{x \rightarrow a} f(x) = f(a)$$

Ex) Find $\lim_{x \rightarrow 2} \frac{x^2 - 1}{x - 1} = \frac{2^2 - 1}{2 - 1}$

$$= \frac{3}{1}$$

Ex) Now, let's take a look at $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$.

*****We need one more useful fact to take this limit:**

If $f(x) = g(x)$ when $x \neq a$ then

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$$

Ex) $\lim_{x \rightarrow 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}} = \lim_{x \rightarrow 1} x+1 = 2$

Sometimes we need to first use some algebra to compute limits:

$$\begin{aligned} \text{Ex) } \lim_{x \rightarrow 1} \frac{\sqrt{x+3}-2}{x-1} &= \frac{\sqrt{x+3}+2}{\sqrt{x+3}+2} = \lim_{x \rightarrow 1} \frac{x+3-4}{(x-1)(\sqrt{x+3}+2)} \\ &= \lim_{x \rightarrow 1} \frac{x-1}{(x-1)(\sqrt{x+3}+2)} \\ &= \lim_{x \rightarrow 1} \frac{1}{\sqrt{x+3}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4} \end{aligned}$$

$$\begin{aligned} \text{Ex) } \lim_{t \rightarrow 0} \left(\frac{1}{t} - \frac{1}{t^2+t} \right) &= \lim_{t \rightarrow 0} \left(\frac{1}{t} - \frac{1}{t(t+1)} \right) = \lim_{t \rightarrow 0} \left(\frac{t+1}{t(t+1)} - \frac{1}{t(t+1)} \right) \\ &= \lim_{t \rightarrow 0} \left(\frac{t+1-1}{t(t+1)} \right) = \lim_{t \rightarrow 0} \left(\frac{t}{t(t+1)} \right) \\ &= \lim_{t \rightarrow 0} \left(\frac{1}{t+1} \right) = \frac{1}{0+1} = 1 \end{aligned}$$

Let's take a look at one more example:

$$\text{Ex) } \lim_{x \rightarrow 0} \frac{x}{|x|} \sin x$$

$$\begin{aligned} x > 0 & \quad \frac{x}{x} \sin x = \sin x \\ x < 0 & \quad \frac{x}{-x} \sin x = -\sin x \end{aligned}$$

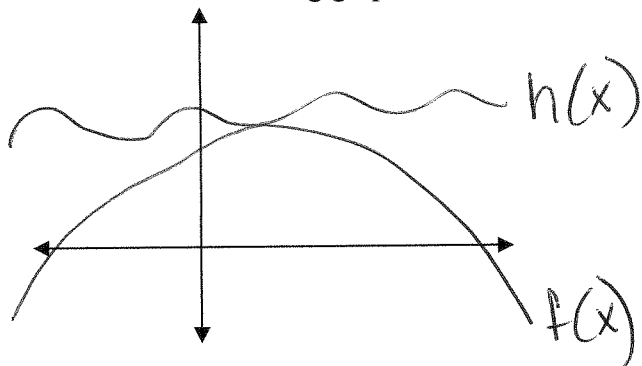
$$\lim_{x \rightarrow 0^+} \frac{x}{|x|} \sin x = \lim_{x \rightarrow 0^+} \sin x = 0$$

$$\lim_{x \rightarrow 0^-} \frac{x}{|x|} \sin x = \lim_{x \rightarrow 0^-} -\sin x = -0 = 0$$

$$\therefore \lim_{x \rightarrow 0} \frac{x}{|x|} \sin x = 0$$

THE SQUEEZE THEOREM (sometimes called The Sandwich Theorem)

Consider the following graph:



If $f(x) \leq h(x)$ when x is near a (except, possibly at a), and the limits of f and h exist at a , then

$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} h(x)$$

Now, what if $f(x) \leq g(x) \leq h(x)$? Then

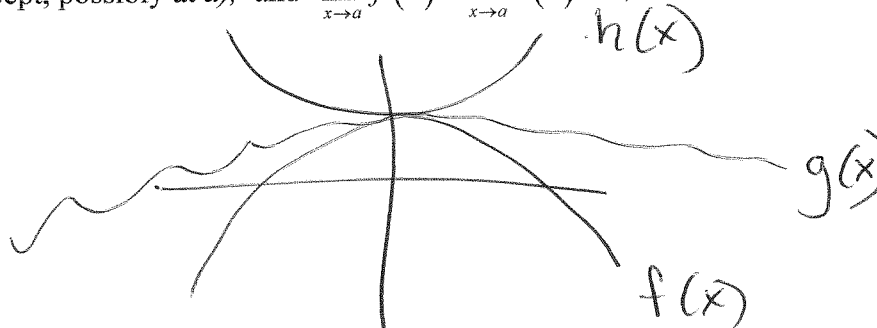
$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x) \leq \lim_{x \rightarrow a} h(x)$$

Now, what if $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x)$?

SQUEEZE THEOREM

If $f(x) \leq g(x) \leq h(x)$ when x is near a (except, possibly at a), and $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$, then

$$\lim_{x \rightarrow a} g(x) = L$$



Ex) #26 p115 Use the squeeze theorem to show that $\lim_{x \rightarrow 0} \sqrt{x^3 + x^2} \sin\left(\frac{\pi}{x}\right) = 0$

$$\lim_{x \rightarrow 0} \sqrt{x^3 + x^2} = 0 \quad \lim_{x \rightarrow 0} -\sqrt{x^3 + x^2} = 0$$

$$-\sqrt{x^3 + x^2} \leq \sqrt{x^3 + x^2} \sin\left(\frac{\pi}{x}\right) \leq \sqrt{x^3 + x^2} \quad \text{since } -1 \leq \sin\left(\frac{\pi}{x}\right) \leq 1$$

by Squeeze

$$\lim_{x \rightarrow 0} \sqrt{x^3 + x^2} \sin\left(\frac{\pi}{x}\right) = 0$$