2.4 Continuity

We found in section 2.3, that in some instances we could find the limit as x approaches a by simply evaluating the function at a. It turns out that these functions are called **continuous at *a*.** We now look at a more formal definition of continuity.

Def
A function f is <u>continuous</u> at a number a if

(i) $f(a)$ is defined (aka $a \in$ domain(f))

(ii) $\lim\limits_{x \to a} f(x)$ exists

(iii) $\lim\limits_{x \to a} f(x) = f(a)$

So, to show that a function is continuous at a number a, we just show that i, ii, iii hold.

Ex) #10 p 126 Use the definition of continuity and the properties of limits to show that $f(x) = x^2 + \sqrt{7-x}$ is continuous at $x = 4$.

i. $f(4) = 4^2 + \sqrt{7-4} = 16 + \sqrt{3} \approx 17.732\ldots$

ii. $\lim\limits_{x \to 4} x^2 + \sqrt{7-x} = \lim\limits_{x \to 4} x^2 + \lim\limits_{x \to 4} \sqrt{7-x}$

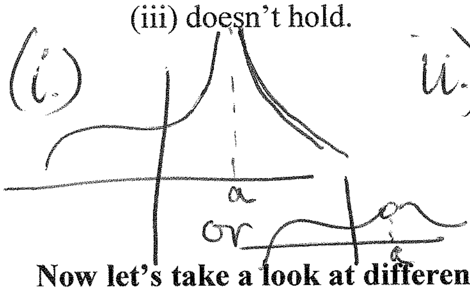
$= \left(\lim\limits_{x \to 4} x\right)^2 + \sqrt{\lim\limits_{x \to 4}(7-x)} = \left(\lim\limits_{x \to 4} x\right)^2 + \sqrt{\lim\limits_{x \to 4} 7 - \lim\limits_{x \to 4} x}$

$= 4^2 + \sqrt{7-4} = 16 + \sqrt{3}$

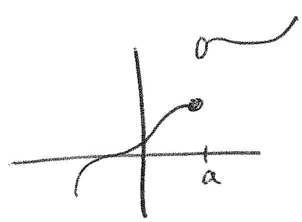
iii. $f(4) = 16 + \sqrt{3} = \lim\limits_{x \to 4} f(x)$ $\therefore$ $f(x)$ is continuous at $x = 4$.

Ex) Sketch examples of a discontinuous function for which (i) doesn't hold; (ii) doesn't hold; (iii) doesn't hold.

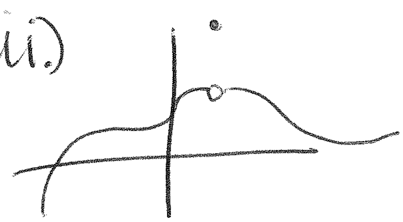
(i.)



ii.)

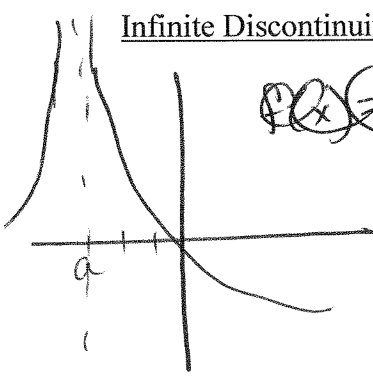


iii.)

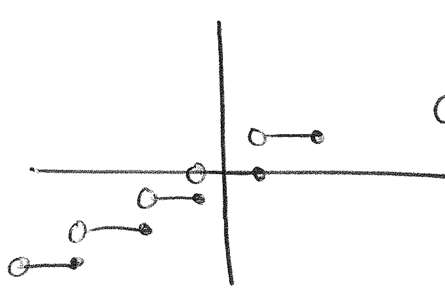


Now let's take a look at different types of DISCONTINUITY.

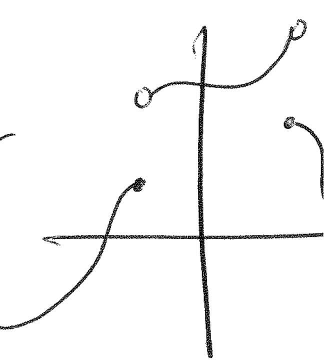
Infinite Discontinuity



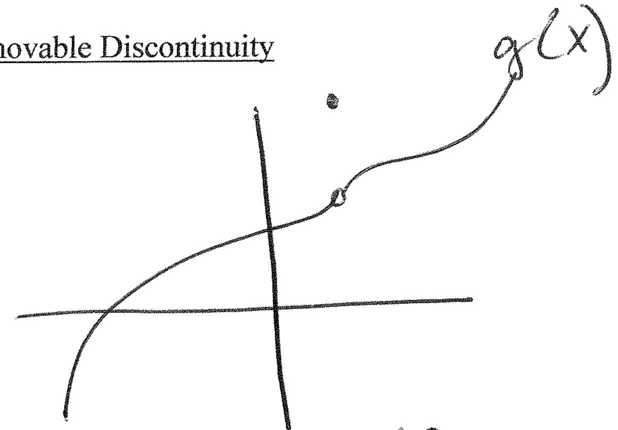
Jump Discontinuity



or



Removable Discontinuity



$$f(x)=\begin{cases} x^3+5x & \text{if } x\neq 2 \\ 20 & \text{if } x=20 \end{cases}$$

Def

A function *f* is <u>continuous from the right at a number *a*</u> if $\lim_{x\to a^+} f(x)=f(a)$

A function *f* is <u>continuous from the left at a number *a*</u> if $\lim_{x\to a^-} f(x)=f(a)$

Thm

If *f* and *g* are continuous at a number *a* and *c* is a constant, then the following are continuous at *a* too.

$f+g$ $\qquad fg$

$f-g$ $\qquad \frac{f}{g}$ if $g(a)\neq 0$

cf

Proof: of $f\cdot g$

Since f & g are continuous at a we know $\lim_{x\to a} f(x)=f(a)$ and $\lim_{x\to a} g(x)=g(a)$

$$\cancel{B}\ \lim_{x\to a} f(x)\cdot g(x)=\lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$$
$$=f(a)\cdot g(a)$$

Thm 7 pg 122 *Luckily, it turns out that most of our "familiar" functions are continuous on their domain. These types of functions are continuous on their domains:*

oots, trig, nverse trig, xponential, logarithmic

Remember Tan undefined at $\pi/2, 3\pi/2, 5\pi/2, \ldots$

polynomials $P(x)=a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\cdots a_2x^2+a_1x+$

rational functions

$\hookrightarrow$ Proof: ~~R(x)~~ $\lim_{x\to a} P(x)=\lim_{x\to a} a_nx^n+\lim_{x\to a} a_{n-1}x^{n-1}+\cdots+\lim_{x\to a} a_1x+\lim_{x\to a} a_0$

$$=a_n\lim_{x\to a} x^n+a_{n-1}\lim_{x\to a} x^{n-1}+\cdots a_1\lim_{x\to a} x+a_0$$
$$=a_na^n+a_{n-1}a^{n-1}+\cdots a_1a+a_0$$
$$=P(a)$$

rational Proof

$$R(x)=\frac{P(x)}{Q(x)}\to\lim R(x)=\frac{\lim P(x)}{\lim Q(x)}=\frac{P(a)}{Q(a)}=R(a)$$

Here is another theorem involving continuity:

Thm If f is continuous at b & $\lim_{x\to a} g(x)=b$, then $\lim_{x\to a} f(g(x))=f(b)$

a.k.a.

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$$

~~**Proof:**~~

Ex) #28 p127 Use continuity to evaluate $\lim_{x\to 2} \arctan\left(\frac{x^2-4}{3x^2-6x}\right)$

$$\arctan \lim_{x\to 2}\left(\frac{x^2-4}{3x^2-6x}\right)$$

$$\arctan\left(\lim_{x\to 2} \frac{(x+2)\cancel{(x-2)}}{3x\cancel{(x-2)}}\right)$$

$$\arctan\left(\lim_{x\to 2} \frac{x+2}{3x}\right)$$

$$\arctan(4/6) \approx .588$$

Ex)#30 p127 Show that f is continuous on $(-\infty,\infty)$, where $f(x)=\begin{cases}\sin x & \text{if } x<\pi/4\\ \cos x & \text{if } x\ge \pi/4\end{cases}$

By Thm 7 $f(x)$ is continuous ~~from~~ $-\infty < x \le \pi/4$ and $\pi/4 < x < \infty$

$$\lim_{x\to \pi/4} f(x) = \begin{cases}\lim_{x\to \pi/4^-} \sin x\\ \lim_{x\to \pi/4^+} \cos x\end{cases} = \begin{cases}\lim_{x\to \pi/4} \sin x\\ \lim_{x\to \pi/4} \cos x\end{cases} = \begin{cases}\sqrt{2}/2\\ \sqrt{2}/2\end{cases}$$

$\Rightarrow f(x)$ is continuous at $\pi/4$ $\therefore f(x)$ is continuous on $(-\infty,\infty)$.

Ex) Are there values of c and m that make $h(x)=\begin{cases}cx^2 & \text{if } x<1\\ 4 & \text{if } x=1\\ -x^3+mx & \text{if } x>1\end{cases}$ continuous at $x = 1$? Find c and m, or explain why the do not exist.

We want $cx^2=4$ at $x=1$ and $-x^3+mx=4$ at $x=1$

$c\cdot 1=4 \Rightarrow c=4$

$-1^3+m\cdot 1=4 \Rightarrow -1+m=4 \Rightarrow m=5$ so $c=4$ $m=5$

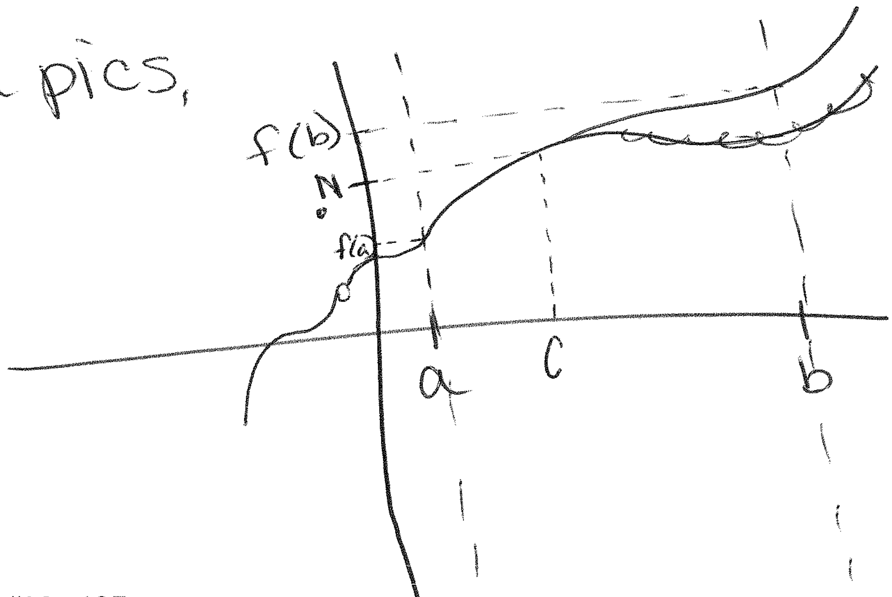
$$h(x)=\begin{cases}4x^2 & \text{if } x<1\\ 4 & \text{if } x=1\\ -x^3+mx & \text{if } x>1\end{cases}$$

The following theorem involving continuity is useful in proving existence of roots of a function.

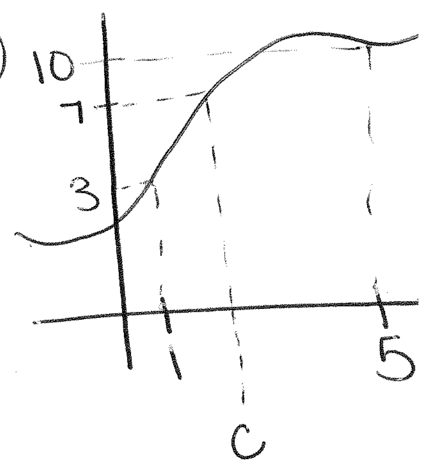
<u>Intermediate Value Theorem (IVT)</u>

Suppose that f is continuous on the closed interval $[a, b]$, and let N be any number between $f(a)$ and $f(b)$ such that $f(a) \neq f(b)$. Then, $\exists$ a number $c \in (a,b) : f(c) = N$

In pics,



ex.



Ex) #38 p127

Use the IVT to show that there is a root of the given equation in the specified interval.

$\sqrt[3]{x} = 1 - x$; on $(0, 1)$

$\sqrt[3]{x} - 1 + x = 0$

Let $f(x) = \sqrt[3]{x} - 1 + x$ (by Thm 7 $f(x)$ is cont on $[0,1]$)

~~Root~~ $f(0) = \sqrt[3]{0} - 1 + 0 = -1$

$f(1) = \sqrt[3]{1} - 1 + 1 = 1$

Thus, $f(0) < 0 < f(1)$ (letting $N = 0$)

By IVT $\exists$ a number $c \in (0,1) : f(c) = 0$