4.5

Note Title 10/25/2007

Indeterminate Forms

ex $\lim_{x \to 1} \frac{\ln x}{x-1}$

can't just do $\lim_{x \to 1} \frac{\ln(1)}{1-1} = \frac{0}{0}$
because of division by zero.
but there's no way to simplify either... so what to do??

Anytime we have a limit in the form
$\lim_{x \to a} \frac{f(x)}{g(x)}$

where $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$
we call it an <u>indeterminate form of type $\frac{0}{0}$</u>

(note: the limit may or may not exist.)

ex. $\lim_{x \to \infty} \frac{\ln x}{x-1}$

note as $x \to \infty$, $\ln x \to \infty$ and $x-1 \to \infty$

called <u>indeterminate form of type $\frac{\infty}{\infty}$</u>

L'Hospital's Rule

We need a way to evaluate these limits, use L'Hospital's Rule.

If f & g are differentiable & $g' \neq 0$ near a and $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = 0$

then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$

Also, if $\lim\limits_{x\to a} f(x) = \infty$ and $\lim\limits_{x\to a} g(x) = \infty$

then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$

(Assuming $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$ exists.)

Proof: The case where $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$

$$\lim_{x\to a}\frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} = \frac{\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}{\lim_{x\to a}\frac{g(x)-g(a)}{x-a}}$$

$$= \lim_{x\to a}\frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} = \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}$$

$$= \lim_{x\to a}\frac{f(x)}{g(x)}$$

$\therefore \lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f(x)}{g(x)}$ under the given conditions.

ex. find $\lim_{x\to 1}\frac{\ln x}{x-1}$

since $\lim_{x\to 1}\ln x = \ln 1 = 0$ and

$\lim_{x\to 1} x-1 = 1-1 = 0$

must show/check this!

we can use L'Hospital's

$$\lim_{x\to 1}\frac{\ln x}{x-1} = \lim_{x\to 1}\frac{\frac{d}{dx}\ln x}{\frac{d}{dx}x-1} = \lim_{x\to 1}\frac{1/x}{1} = 1$$

ex. $\lim_{x\to\infty}\frac{\ln x}{x-1}$ $\quad \lim_{x\to\infty}\ln x = \infty \quad \lim_{x\to\infty} x-1 = \infty$

$$\lim_{x\to\infty}\frac{\ln x}{x-1} = \lim_{x\to\infty}\frac{1/x}{1} = 0$$

ex. $\lim_{t\to 0} \frac{e^{3t}-1}{t}$

$\lim_{t\to 0} e^{3t}-1 = e^0 - 1 = 0$

$\lim_{t\to 0} t = 0$

can use L'Hospital's

$$\lim_{t\to 0}\frac{e^{3t}-1}{t} = \frac{\lim_{t\to 0} 3e^{3t}}{\lim_{t\to 0} 1} = 3$$

$\boxtimes$

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Indeterminate form of type $0\cdot\infty$ (or $0\cdot -\infty$)

if we have $f\cdot g$ and $f\to 0$, $g\to\infty$ as $x\to a$

then $1/g \to 0$ as $x\to a$

$\Rightarrow \frac{f}{1/g} \to \frac{0}{0}$ as $x \to a$

but $\frac{f}{1/g} = f\cdot g$ (or $\frac{g}{1/f} = g\cdot f$)

So when we have indeterminate form of type $0\cdot\infty$ we can rewrite it as $\frac{0}{0}$

ex. $\lim_{x \to -\infty} x^2 e^x$

$\lim_{x \to -\infty} x^2 = \infty$ $\quad$ $\lim_{x \to -\infty} e^x = 0$

$\Rightarrow \lim_{x \to -\infty} \frac{1}{e^x} = \infty$

$x^2 e^x = \frac{x^2}{1/e^x} \Rightarrow \lim_{x \to -\infty} x^2 e^x = \lim_{x \to -\infty} \frac{x^2}{1/e^x} = \frac{x^2}{e^{-x}}$

but $x^2 \to \infty$, $1/e^x \to \infty$ as $x \to -\infty$
so we can apply L'Hospitals

$= \lim_{x \to -\infty} \frac{2x}{-e^{-x}}$

But now $\lim_{x \to -\infty} 2x = -\infty$ &
$\lim_{x \to -\infty} \frac{-1}{e^x} = -\infty$ so apply L'Hospitals
again

$= \lim_{x \to -\infty} \frac{2}{e^{-x}} = \lim_{x \to -\infty} 2e^x = 0$

Indeterminate differences

$\infty - \infty$

idea: rewrite as division

ex. $\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$

$= \lim_{x\to 1}\left(\frac{x-1}{(x-1)\ln x} - \frac{\ln x}{(x-1)\ln x}\right)$

$= \lim_{x\to 1}\left(\frac{x-1-\ln x}{(x-1)\ln x}\right)$

as $x\to 1$, $x-1-\ln x \to 0$ and $(x-1)\ln x \to 0$

so L'Hospitals gives

$= \lim_{x\to 1}\left(\frac{1-1/x}{(x-1)1/x + \ln x}\right) = \lim_{x\to 1} \frac{x-1}{x-1+x\ln x}$

as $x\to 1$, $x-1\to 0$ and $x-1+x\ln x \to 0$

so L'Hospitals again

$= \lim_{x\to 1} \frac{1}{1+x\,1/x+\ln x} = \frac{1}{1+1+0} = 1/2$

$\blacksquare$

Indeterminate Powers

3 types for $\lim_{x\to a}[f(x)]^{g(x)}$

0^0 (conflict $0^n=0$ but $n^0=1$)

∞^0 (conflict $\infty^n=\infty$ but $n^0=1$)

1^∞ (conflict $n^\infty=\infty$ but $1^n=1$)

General Approach, let $y=[f(x)]^{g(x)}$ then

$\ln y = g(x)\ln f(x)$

use L'Hospitals to find $\lim_{x\to a}\ln y$

then use

$$\lim_{x\to a} y = \lim_{x\to a} e^{\ln y} = e^{\lim_{x\to a}\ln y}$$

Ex. $\lim_{x\to 0^+}(\tan 2x)^x$ $\quad$ $\lim_{x\to 0^+}\tan 2x=0$

$\lim_{x\to 0^+} x=0$ $\quad$ 0^0

$y=(\tan 2x)^x$

$\ln y = x\ln\tan 2x$

$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+} x\ln\tan 2x = \lim_{x\to 0^+}\dfrac{\ln\tan 2x}{1/x}$

$\ln(\tan(2x))\to-\infty$ & $1/x\to\infty$ as $x\to 0^+$

So L'Hospital's

$$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\frac{\frac{1}{\tan 2x}\cdot\sec^2 2x\cdot 2}{-1/x^2} = \lim_{x\to 0^+} -\frac{\cos 2x}{\sin 2x}\cdot\frac{1}{\cos^2 2x}\cdot 2x^2$$

$$= \lim_{x\to 0^+}\frac{-2x^2}{\sin 2x\cos 2x}$$

$-2x^2\to 0$ & $\sin 2x\cos 2x\to 0$ as $x\to 0^+$

$$= \lim_{x\to 0^+} \frac{-4x}{-2\sin^2 2x + 2\cos^2 2x}$$

$$= \frac{-4(0)}{-2\sin^2(2\cdot 0) + 2\cos^2(2\cdot 0)} = \frac{0}{0+2\cdot 1} = 0$$

$$\Rightarrow \lim_{x\to 0^+} \ln y = 0$$

and $\lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^{\lim_{x\to 0^+} \ln y} = e^0 = 1$

So $\lim_{x\to 0^+} (\tan 2x)^x = 1$

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ex. $\lim_{x\to\infty} x^{\frac{\ln 2}{1+\ln x}}$ $\quad x\to\infty$ & $\frac{\ln 2}{1+\ln x}\to 0$ as $x\to\infty$

type ∞^0

let $y = x^{\frac{\ln 2}{1+\ln x}} \Rightarrow \ln y = \frac{\ln 2}{1+\ln x}\ln x$

$\ln x\to\infty$ & $\frac{1+\ln x}{\ln 2}\to\infty$ as $x\to\infty$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln 2}{1+\ln x}\ln x = \lim_{x\to\infty} \frac{\ln x}{\frac{1+\ln x}{\ln 2}}$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{1/x}{\ln 2/x} = \lim_{x\to\infty} \ln 2 = \ln 2$$

$$\Rightarrow \lim_{x\to\infty} \ln y = \ln 2$$

$$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^{\lim_{x\to\infty}\ln y} = e^{\ln 2} = 2$$

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