4.1 Related Rates

- The idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which we may know more information about).

Suppose we want to compute the rate of change of the radius of a circle with respect to time. Also suppose we know the rate at which the area of the circle is changing with respect to time.

- We will find an equation that relates the quantities, then differentiate with respect to time using the chain rule.

So we find an equation that relates the area of a circle with its radius ($A = \pi r^2$). Then, we take the derivative of both sides with respect to time, t, using the chain rule ($\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$).

Steps in Solving Related Rates Problems

1. Read the problem carefully. If appropriate, draw a picture that represents the situation.

2. Assign variables to quantities and write down what is known and what is unknown.

3. Write an equation that relates quantities. (Often used: Pythagorean Theorem, trigonometric relationships, similar triangles, formulas for volume and area, etc.)

4. Differentiate the equation with respect to time (we use the chain rule here)

5. Substitute known values in the resulting equation.

6. Solve for the unknown.

7. Answer the question (make sure you answer the question being asked!!!)

Ex 1) You are blowing a bubble with bubble gum and can blow air into the bubble at a rate of 3 in^3/s.

a. How fast is the radius increasing with respect to time when $r = 3$ inches?

$\frac{dV}{dt} = 3 \ in^3/s \qquad r = 3 \ in \quad \frac{dr}{dt} = ?$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow 3 = 4\pi (3)^2 \cdot \frac{dr}{dt}$

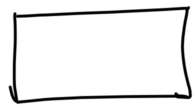
$\Rightarrow dr/dt = 3/36\pi \ in/s$

b. Suppose you increase your effort when $r = 3$ inches and begin to blow in air at a rate of 4 in^3/s. How fast is the radius increasing now?

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow 4 = 4\pi (3)^2 \cdot \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{1}{9\pi} \ in/sec$

Ex 2) #4 p267 The length of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at a rate of 3 cm/s. When the length is 20cm and the width is 10cm, how fast is the area of the rectangle increasing?



$L = 20 \quad \frac{dL}{dt} = 8$

$w = 10 \quad \frac{dw}{dt} = 3$

$\frac{dA}{dt} = ?$

$A = L * w$

$$\frac{dA}{dt} = L * \frac{dw}{dt} + w \frac{dL}{dt}$$

$$\frac{dA}{dt} = 20 * 3 + 10 * 8$$

$$= 60 + 80 = 140 \text{ cm}^2/\text{s}$$

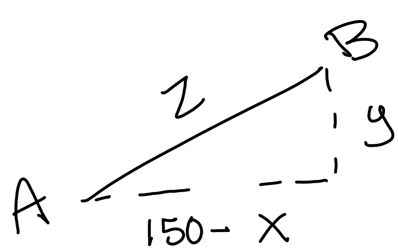
Ex 3) #10 p267 At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 pm?

$\frac{dx}{dt} = 35 \rightarrow$ $\quad 25 = \frac{dy}{dt} \uparrow$

ship A ——— ship B

150 km

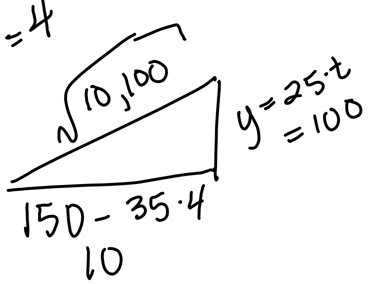
$\frac{dD}{dt} = ?$ at $t = 4$



$$z^2 = (150 - x)^2 + y^2$$

$$2z \frac{dz}{dt} = 2(150 - x)\left(-\frac{dx}{dt}\right) + 2y \frac{dy}{dt}$$

at $t = 4$

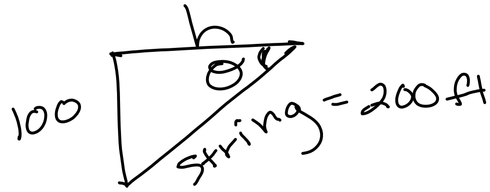


$$\frac{dz}{dt} = \frac{1}{z}\left[(150 - x)\left(-\frac{dx}{dt}\right) + y \frac{dy}{dt}\right]$$

$$= \frac{1}{\sqrt{10{,}100}}(-10 \cdot 35 + 100 \cdot 25)$$

$$= 2150/\sqrt{10{,}100} \approx 21.39$$

Ex 4) #26 p268 A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string have been let out?



$\frac{d\theta}{dt} = ?$ $\frac{dh}{dt} = 8$ ft/s

$\tan\theta = \frac{100}{h}$

$\theta = \tan^{-1}\frac{100}{h}$

$\frac{d\theta}{dt} = \frac{d\theta}{dh}\cdot\frac{dh}{dt}$

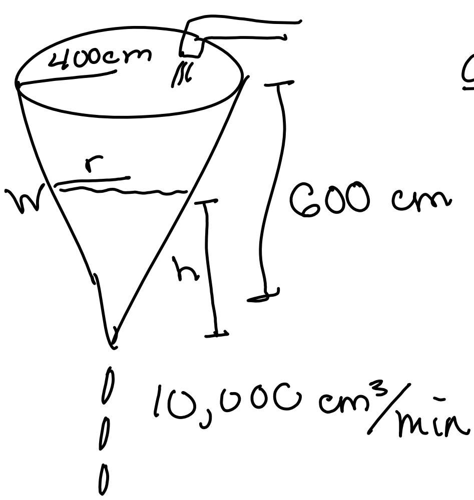
$200^2 = 100^2 + h^2$

$\Rightarrow h = 173.205$

$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{100}{h}\right)^2}\cdot\frac{-100}{h^2}\cdot\frac{dh}{dt}$

$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{100}{h}\right)^2}\cdot\frac{-100}{h^2}\cdot 8 = -.02$ decreasing .02

Ex 5) #22 p268 Water is leaking out of an inverted conical tank at a rate of 10,000 cm^3/min at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the height of the water is 2 m, find the rate at which water is being pumped into the tank.



$\frac{dW}{dt} = 20$ cm/min when $h = 200$ cm

$V = \frac{1}{3}\pi r^2 h$

by similiar $\Delta \Rightarrow \frac{400}{600} = \frac{r}{h}$

$\Rightarrow \frac{r}{h} = \frac{2}{3} \Rightarrow r = \frac{2}{3}h$

so $V = \frac{1}{3}\pi\left(\frac{2}{3}h\right)^2\cdot h$

$V = \frac{1}{3}\pi\frac{4}{9}h^3$

$V = \frac{4}{27}\pi h^3$

$$V = \frac{4}{27}\pi h^3$$

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{4}{9}\pi h^2 \cdot \frac{dh}{dt}$$

$$\frac{dV}{dt}(200) = \frac{4}{9}\pi(200)^2 \cdot 20 = \frac{3200000\pi}{9}$$

$$\frac{dV}{dt} = In - 10,000$$

$$\frac{3200000\pi}{9} = In - 10,000$$

$$In = \frac{3200000\pi}{9} + 10,000$$

$$\approx 1.2701 \times 10^6$$