4.3 Derivatives and the Shapes of Curves

Mean Value Theorem

If f is differentiable on $[a, b]$ then there exist a number c between a and b such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

or

$$f(b)-f(a) = f'(c)(b-a)$$

For a real-life application to the MVT, consider the position function $s = f(t)$. Then the average velocity between times $t = a$ and $t = b$ is $\dfrac{f(b)-f(a)}{b-a}$. The MVT says that at some point on the interval (a, b) the instantaneous velocity, $f'(c)$ is equal to the average velocity.

> Say you are going to drive to Chicago (420 miles) away. Suppose you leave at noon and pull into Chicago at 5:30 pm. Given this information, a police officer could bust you for speeding according to the MVT. Why? $(0,0)$ $(5.5, 420)$

$$f'(c) = \frac{420-0}{5.5-0} = 76.\overline{36}$$

so avg velocity was $76.\overline{36}$ mph & by MVT at some time c you were going $76.\overline{36}$

Now we recall some facts from section 2.9.

> If $f'(x) > 0$ on an interval, then f is <u>increasing</u> on that interval.
>
> If $f'(x) < 0$ on an interval, then f is <u>decreasing</u> on that interval.

Ex) Find the intervals where g is increasing and where it is decreasing, given $g(x) = x^4 + \frac{8}{3}x^3 - 6x^2 + 7$.

$$g'(x) = 4x^3 + 8x^2 - 12x = 0$$

$$4x(x+3)(x-1) = 0$$

- - - | ++++ | - - - | +++

-3, 0, 1

inc. $(-3, 0) \cup (1, \infty)$

dec. $(-\infty, -3) \cup (0, 1)$

The First Derivative Test

Suppose c is a critical number of a continuous function f.

a. If f' changes from + to – at c, then f has a local max at c.

b. If f' changes from – to + at c, then f has a local min at c.

c. If f' does not change sign at c, then there is no local maximum or minimum at c.

Ex)Use the First Derivative Test to state where the local maxima are found in the previous example.

local min

$x = -3, 1$

local max

$x = 0$

Now we recall some more facts from section 2.9.

If $f''(x) > 0$ on an interval, then (f' is increasing) and f is concave up on that interval.

If $f''(x) < 0$ on an interval, then (f' is decreasing) and f is concave down on that interval.

Also, the point at which f changes concavity is called an inflection point.

The Second Derivative Test

Suppose f'' is continuous near c.

a. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local min at c.

b. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local max at c.

*** The second derivative test is inconclusive if $f''(c) = 0$.

Ex) Use the second derivative test to state where the local maxima are found in the previous example.

$g'(x) = 0 \Rightarrow x = -3, 0, 1$

$g''(x) = 12x^2 + 16x - 12$

$g''(0) = -12 \leftarrow$ local max

$g''(1) = 16 \leftarrow$ local min

$g''(-3) = 48 \leftarrow$ local min

local max at $x = 0$

Ex) #24 p287 Let $B(x) = 3x^{2/3} - x$.

a. Find the intervals of increase or decrease.
b. Find local maximum and minimum values.
c. Find the intervals of concavity and the inflection points.
d. Use the information from a-c to sketch a graph of the function.

a) $B' = 2x^{-1/3} - 1$

$\frac{2}{\sqrt[3]{x}} = 1 \qquad 2 = \sqrt[3]{x} \qquad x = 0$

$x = 8$

$+ + + +$

$- - - -\ 0 \qquad 8\ - - - -$

inc $(0,8)$

dec $(-\infty,0) \cup (8,\infty)$

b) local min @ $x = 0$ local min is 0

local max @ $x = 8$ local max is 4

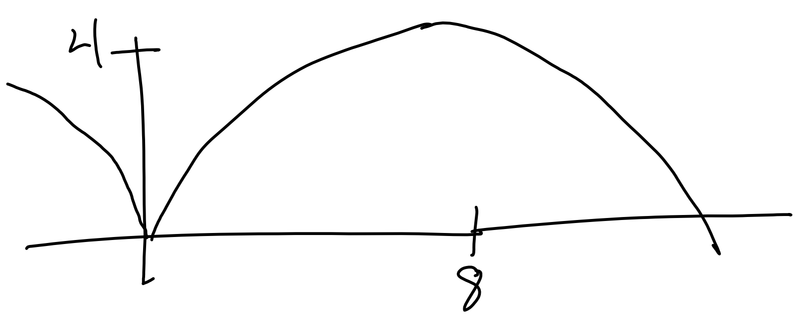
c) $B'' = -\frac{2}{3}x^{-4/3} = \frac{-2}{3\sqrt[3]{x^4}} \qquad x = 0$

concave down $(-\infty,0) \cup (0,\infty)$

no inflection point

$- - -\ 0\ - - -$

d)



Ex) Let $f(x) = \frac{x^2}{(x-2)^2}$

a. Find any vertical and horizontal asymptotes.
b. Find the intervals of increase or decrease.
c. Find local maximum and minimum values.
d. Find the intervals of concavity and the inflection points.
e. Use the information from a-d to sketch a graph of the function.

a) $\frac{x^2}{(x-2)^2}$ is undefined at $x=2$

$\lim_{x \to 2} \frac{x^2}{(x-2)^2} \to \frac{4}{0} \to \infty$ so $x=2$ is VA

$\lim_{x \to \infty} \frac{x^2}{(x-2)^2} = \lim_{x \to \infty} \frac{x^2}{x^2-4x+4} = \lim_{x \to \infty} \frac{1}{1-4/x+4/x^2} = 1$

so $y=1$ is a HA ($\lim_{x \to -\infty} f(x)$ same)

b) $f' = \frac{(x-2)^2 \cdot 2x - x^2 2(x-2)}{(x-2)^4}$

$= \frac{2x(x-2)[x-2-x]}{(x-2)^4} = \frac{-4x}{(x-2)^3}$ $\quad x = 0, 2$

$- - - \quad 0 \quad +++ \quad 2 \quad - - -$

inc $(0,2)$ dec $(-\infty, 0) \cup (2, \infty)$

c) local min @ $x=0$ (undefined at $x=2$)

d) $f'' = \frac{-4(x-2)^3 + 12x(x-2)^2}{(x-2)^6} = \frac{4(x-2)^2(-x+2+3x)}{(x-2)^6} = \frac{8(x+1)}{(x-2)^4}$

$f''=0 \Rightarrow x=-1$
$x=2$

$- - - \quad -1 \quad +++ \quad 2 \quad +++$

concave down $(-\infty, -1)$
concave up $(-1, 2) \cup (2, \infty)$
inflection pt @ $x=-1$

e)