5.2

Note Title 11/27/2007

Defn.

If f is a continuous function defined for $a \le x \le b$, we divide the interval $[a,b]$ into n subintervals of equal width $\Delta x = (b-a)/n$. We let $x_0 (=a), x_1, x_2, \ldots, x_n(=b)$ right endpoints of the corresponding subinterval. The definite integral of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

if left endpoints $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$

Notation

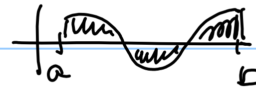
$$\underbrace{\int}_{\text{integral sign}}\overbrace{{}^{b}}^{\text{upper limit of integration}}_{\underbrace{a}_{\text{lower limit of integration}}} \underbrace{f(x)}_{\text{integrand}}\,dx$$

$\sum_{i=1}^{n} f(x_i)\,\Delta x$ is called a <u>Riemann sum</u>

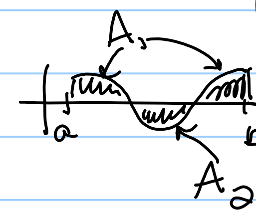
$\int_a^b f(x)\,dx$ can be interpreted as the area under the curve from a to b.

(if $f(x) > 0$)

if $f(x)$ takes on positive & negative values

ex. 

then $\int_a^b f(x)\,dx$ is the net area

$$\int_a^b f(x)\,dx = A_1 - A_2$$ where 

Ex. #18 Express $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{e^{x_i}}{1+x_i} \Delta x$ $[1,5]$ as a definite integral

$$\int_1^5 \frac{e^x}{1+x}\,dx$$

Evaluating Integrals

Formulas for sums

1. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
2. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
3. $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$
4. $\sum_{i=1}^{n} c = nc$
5. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$
6. $\sum_{i=1}^{n} (a_i+b_i) = \sum a_i + \sum b_i$
7. $\sum (a_i - b_i) = \sum a_i - \sum b_i$

Ex. find Riemann sum for $f(x)=x^2+x-2$; $0 \le x \le 4$ with 8 subintervals using right endpoints.

$$\sum_{i=1}^{8}(x_i^2+x_i-2)\Delta x \quad , \quad \Delta x = \frac{4-0}{8} = \frac{1}{2}$$

$$= \frac{1}{2}(-1.25+0+1.75+4+6.75+10+18)$$

$$= \frac{1}{2}(39.25) = 19.625$$

Evaluate $\int_0^4 x^2+x-2\,dx$

now n subintervals so $\Delta x = \frac{4-0}{n} = \frac{4}{n}$

$$\Rightarrow x_1 = 4/n,\ x_2 = 8/n,\ x_3 = 12/n, \dots, x_i = 4i/n$$

$$\Rightarrow \int_0^4 f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}(x_i^2+x_i-2)\,4/n$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\left(\frac{4i}{n}\right)^2+\left(\frac{4i}{n}\right)-2\right)4/n$$

$$= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\left(\frac{16i^2}{n^2}+\frac{4i}{n}-2\right) \quad \text{by } \#5$$

$$= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\frac{16i^2}{n^2}+\frac{4}{n}\sum_{i=1}^{n}\frac{4i}{n}-\frac{4}{n}\sum_{i=1}^{n}2 \quad \text{by } \#6 \,\&\, \#7$$

$$= \lim_{n\to\infty}\frac{64}{n^3}\sum_{i=1}^{n}i^2+\frac{16}{n^2}\sum_{i=1}^{n}i-\frac{4}{n}\cdot 2n \quad \text{by } \#4,5$$

$$= \lim_{n\to\infty}\frac{64}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}+\frac{16}{n^2}\frac{n(n+1)}{2}-8$$

$$= \frac{64}{6}\lim_{n\to\infty}\frac{n(n+1)(2n+1)}{n^3}+8\lim_{n\to\infty}\frac{n(n+1)}{n^2}-8 \quad \text{by } \#1 \,\&\, \#2$$

$$= \frac{64}{6}\cdot 2+8\cdot 1-8 = 21.\overline{3}$$

Midpoint Rule

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\Delta x$$

midpt of each subinterval

where $\Delta x = \frac{b-a}{n}$ and $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$

Ex. #12 $\int_1^5 x^2 e^{-x}\,dx$, $n=4$

endpts of subintervals are 1, 2, 3, 4, 5

so $\bar{x}_i = 1.5, 2.5, 3.5, 4.5$

$$\int_1^5 x^2 e^{-x}\,dx \approx \sum_{i=1}^{4} \bar{x}_i^{\,2} e^{-\bar{x}_i}\,\frac{4}{4}$$

$$= 1.5^2 e^{-1.5} + 2.5^2 e^{-2.5} + 3.5^2 e^{-3.5} + 4.5^2 e^{-4.5}$$

$$\approx 1.6099$$

Properties of Integrals

1. $\int_a^b c\,dx = c(b-a)$ where c is a constan

2. $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

3. $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$

4. $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

5. $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

6. If $f(x) \geq 0$ for $a \leq x \leq b$ then $\int_a^b f(x)\,dx \geq 0$

7. If $f(x) \geq g(x)$ for $a \leq x \leq b$ then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

8. If $m \leq f(x) \leq M$ for $a \leq x \leq b$ then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$