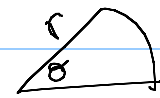
Note Title 2/9/2008

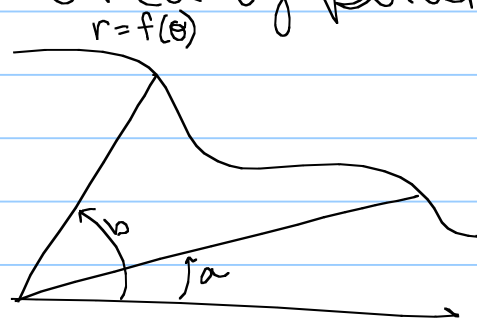
H2

Arc lengths & Polar Coordinates

Area of a sector is $A = \frac{1}{2} r^2 \theta$

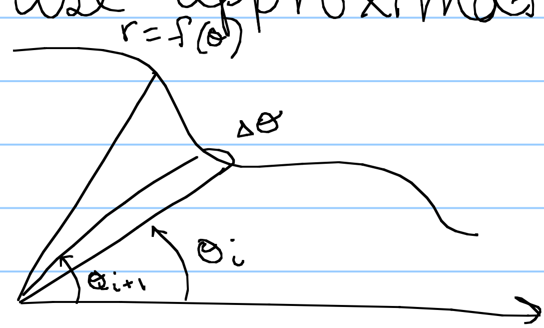


To find area of polar region



notice θ goes from a to b

We use approximating sectors.



$$A = \int_a^b \frac{1}{2} r^2 \, d\theta$$

ex. #6 find area of shaded region

$r = 1 + \sin\theta$

$$A = \int_{\pi/2}^{\pi} \frac{1}{2} (1+\sin\theta)^2 \, d\theta$$

$$= \frac{1}{2} \int_{\pi/2}^{\pi} 1 + 2\sin\theta + \sin^2\theta \, d\theta$$

$$= \frac{1}{2} \Big[\theta - 2\cos\theta\Big]_{\pi/2}^{\pi} + \frac{1}{2} \int_{\pi/2}^{\pi} \sin^2\theta \, d\theta$$

by #63

$$= \frac{1}{2}\left[\pi + 2 - \frac{\pi}{2} + 0\right] + \frac{1}{2}\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_{\pi/2}^{\pi}$$

$$= \frac{\pi}{2} + 1 - \frac{\pi}{4} + \frac{1}{2}\left[\frac{1}{2}\pi - 0 - \frac{\pi}{4} + 0\right]$$

$$= \frac{\pi}{2} + 1 \cancel{- \frac{\pi}{4}} + \cancel{\frac{\pi}{4}} - \frac{\pi}{8}$$

$$= \frac{3\pi}{8} + 1$$

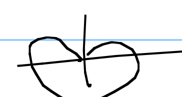
Graph it To find a and b.

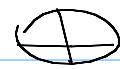
Area between two curves.

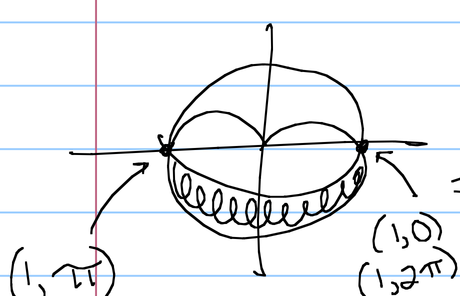
$$A = \int_a^b \frac{1}{2} f^2 \, d\theta - \int_a^b \frac{1}{2} g^2 \, d\theta$$

where $f \geq g \geq 0$ and $0 < b - a \leq 2\pi$

↳ says f is outside, g is inside.

ex. #20 

find area inside $r = 1 + \sin\theta$ and outside $r = 1$ 



$$\int_\pi^{2\pi} \frac{1}{2}(1+\sin\theta)^2 \, d\theta \int_0^\pi \frac{1}{2} \, d\theta$$

$$= \frac{1}{2}\left[\theta - 2\cos\theta + \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_\pi^{2\pi} - \frac{1}{2}\left[\theta\right]_\pi^{2\pi}$$

$$= \frac{1}{2}\left(\cancel{2\pi} - 2 + \pi - 0 \cancel{- \pi} - 2 - \frac{\pi}{2} + 0 \cancel{- 2\pi} \cancel{+ \pi}\right)$$

$$= \frac{1}{2}\left(-4 + \frac{1}{2}\pi\right) = \frac{\pi}{4} - 2$$

Arc Length

for justification see pg. A67

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

is the length of a curve w/ polar equation $r = f(\theta)$, $a \le \theta \le b$.

ex. #36

find length of curve $r = e^{2\theta}$ $\quad 0 \le \theta \le 2\pi$

$$L = \int_0^{2\pi} \sqrt{e^{4\theta} + 4e^{4\theta}}\, d\theta$$

$\hookrightarrow \frac{dr}{d\theta} = 2e^{2\theta}$

$$= \int_0^{2\pi} \sqrt{5e^{4\theta}}\, d\theta$$

$$= \sqrt{5}\int_0^{2\pi} e^{2\theta}\, d\theta$$

let $u = 2\theta$

$\frac{1}{2}du = d\theta$

$$= \frac{\sqrt{5}}{2}\int_0^{2\pi} e^{u}\, du$$

$$= \frac{\sqrt{5}}{2}\left[e^{2\theta}\right]_0^{2\pi} = \frac{\sqrt{5}}{2}\left(e^{4\pi} - 1\right)$$