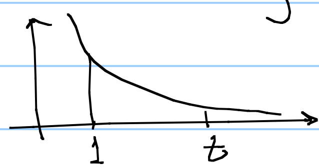
5.10

Note Title 1/11/2008

Improper Integrals

Type I: Infinite Intervals

ex Consider area under curve $y = 1/x^2$ to the right of $x = 1$



$$A(t) = \int_1^t \frac{1}{x^2}\,dx = -x^{-1}\Big|_1^t$$

$$= -\frac{1}{t} - -1 = 1 - \frac{1}{t}$$

$\frac{1}{t} \in (0,1)$ since $t > 1 \Rightarrow A(t) < 1$ no matter how big t gets!

ex Say $t = 10 \Rightarrow A(t) = 1 - \frac{1}{10} = \frac{9}{10}$

$t = 100 \Rightarrow A(t) = 1 - \frac{1}{100} = \frac{99}{100}$

$t = 1000 \Rightarrow A(t) = \frac{999}{1000}$

$\vdots$

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1$$

or

as $t \to \infty$, $\frac{1}{t} \to 0 \Rightarrow 1 - \frac{1}{t} \to 1$

We write

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = 1$$

Definitions

Convergent/Divergent

If $\int_a^\infty f(x)dx$ (or $\int_{-\infty}^b f(x)dx$) exist as finite #s the integrals are said to **converge** (they are convergent)
If they don't, they diverge (divergent)

1.) If $\int_a^t f(x)dx$ exists $\forall\ t \geq a$ then

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

2.) If $\int_t^b f(x)dx$ exists $\forall\ t \leq b$ then

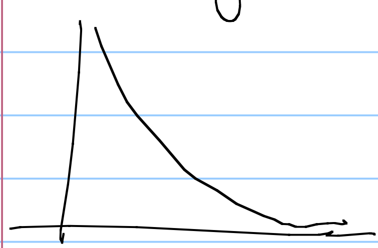
$$\int_{-\infty}^b f(x)dx = \lim_{t\to -\infty} \int_t^b f(x)dx$$

provided limits exist

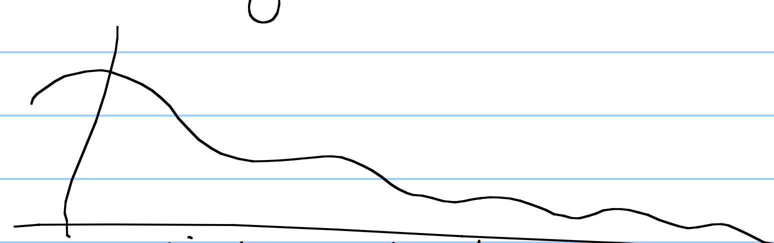
3.) If both $\int_a^\infty f(x)dx$ & $\int_{-\infty} f(x)dx$ converge then

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$$

idea of convergent/divergent



"converges" to x-axis fast enuf that it is finite



"diverges" although it is approaching x-axis it does so so slowly that there's ∞ area below it ex. $\int_0^\infty \frac{x}{(x^2+2)^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{x}{(x^2+2)^2}\,dx$

let $u = x^2+2 \Rightarrow du = 2x\,dx \Rightarrow x\,dx = \frac{1}{2}du$

$= \frac{1}{2}\lim_{t\to\infty}\int_{x=0}^{x=t}\frac{1}{u^2}\,du = \frac{1}{2}\lim_{t\to\infty}\frac{-1}{u}\Big|_{x=0}^{x=t} = \frac{-1}{2}\lim_{t\to\infty}\frac{1}{x^2+2}\Big|_{x=0}^{x=t}$

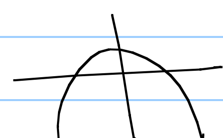
$= \frac{-1}{2}\lim_{t\to\infty}\left(\frac{1}{t^2+2}-\frac{1}{2}\right) = \frac{-1}{2}\cdot -\frac{1}{2} = \frac{1}{4}$

ex. $\int_{-\infty}^0 (2-v^4)\,dv = \lim_{t\to-\infty}\int_t^0 (2-v^4)\,dv$

$= \lim_{t\to\infty}\left(2v-\frac{1}{5}v^5\right)\Big|_t^0 = \lim_{t\to\infty}\left|0-2t+\frac{1}{5}t^5\right|$

$= -\infty$ so $\int_{-\infty}^0 (2-v^4)\,dv$ diverges

see graph



ex. $\int_{-\infty}^\infty x^2 e^{-x^3}\,dx = \int_{-\infty}^0 x^2 e^{-x^3}\,dx + \int_0^\infty x^2 e^{-x^3}\,dx$

$= \lim_{t\to-\infty}\int_t^0 x^2 e^{-x^3}\,dx + \lim_{t\to\infty}\int_0^t x^2 e^{-x^3}\,dx$

let $u = -x^3 \Rightarrow du = -3x^2\,dx \Rightarrow x^2\,dx = \frac{-1}{3}du$

$= \frac{-1}{3}\lim_{t\to-\infty}\int_{x=t}^{x=0} e^u\,du + \frac{-1}{3}\lim_{t\to\infty}\int_{x=0}^{x=t} e^u\,du$

$$= -\frac{1}{3}\left[\lim_{t\to-\infty} e^u \Big|_{x=t}^{x=0} + \lim_{t\to\infty} e^u \Big|_{x=0}^{x=t}\right]$$

$$= -\frac{1}{3}\left[\lim_{t\to-\infty} \frac{1}{e^{x^3}}\Big|_t^0 + \lim_{t\to\infty} \frac{1}{e^{x^3}}\Big|_0^t\right]$$

$$= -\frac{1}{3}\left[\lim_{t\to-\infty}\left(\frac{1}{e^{t^3}} - 1\right) + \lim_{t\to\infty}\left(\frac{1}{e^{t^3}} - 1\right)\right]$$

$$= -\frac{1}{3}\left(\infty + -1\right) = \infty$$

So $\int_{-\infty}^{\infty} x^2 e^{-x^3}\,dx$ diverges.

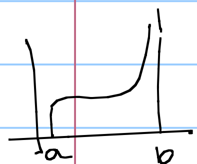
$\blacksquare$

Important Result (for justification see pg 426 ex 4)

$\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p>1$ and divergent if $p\le 1$

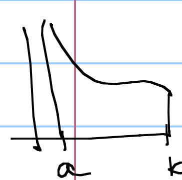
Type II Discontinuous Integrands

1) If f is continuous on $[a,b)$ & discontinuous at b, then



$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

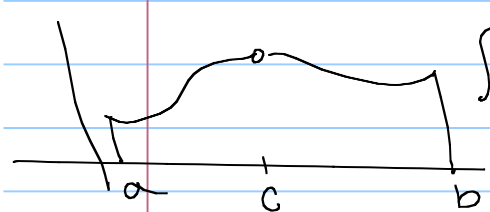
2) If f is continuous on $(a,b]$ & discontinuous at a then



$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$$

assume limits exist

3) If f is discontinuous at c, where $a<c<b$ and $\int_a^c f(x)dx$ & $\int_c^b f(x)dx$ are convergent



$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$= \lim_{t\to c^-}\int_a^t f(x)dx + \lim_{t\to c^+}\int_t^b f(x)dx$$

ex) evaluate $\displaystyle\int_0^1 \frac{\ln x}{\sqrt{x}}dx = \lim_{t\to 0^+}\int_t^1 \frac{\ln x}{\sqrt{x}}dx$

$u = \ln x \qquad dv = x^{-1/2}dx$

$du = \frac{1}{x}dx \qquad v = 2x^{1/2}$

$$= \lim_{t\to 0^+}\int_t^1 \ln x \cdot x^{-1/2}dx = \lim_{t\to 0^+}\left[2\sqrt{x}\ln x\Big|_t^1 - \int_t^1 2x^{-1/2}dx\right]$$

$$= \lim_{t\to 0^+}\left\{0 - 2\sqrt{t}\ln t - \left[4\sqrt{x}\right]_t^1\right\}$$

$$= \lim_{t\to 0^+} \left\{ -2\sqrt{t}\ln t - 4 + 4\sqrt{t} \right\}$$

$= -4$ so $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$ converges

Comparison Theorem

Suppose f & g are continuous & $f(x) \geq g(x) \geq 0$ for $x \geq a$ then

1.) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent

2.) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent

Ex: Determine if $\int_1^\infty \frac{2+e^{-x}}{x}\,dx$ con-/di-verges

$$\frac{2+e^{-x}}{x} = \frac{2}{x} + \frac{e^{-x}}{x} = \frac{2}{x} + \frac{1/e^x}{x} = \frac{2}{x} + \frac{1}{e^x x}$$

$\frac{1}{xe^x} > 0$ since $x \in (1,\infty) \Rightarrow \frac{2}{x} + \frac{1}{xe^x} > \frac{2}{x}$

by $\frac{1}{x}$ diverges $\Rightarrow 2\left(\frac{1}{x}\right)$ diverges

$\Rightarrow \frac{2}{x} + \frac{1}{xe^x}$ diverges $\Rightarrow \frac{2+e^{-x}}{x}$ diverges