5.7

Note Title 1/8/2008

Trig Integrals

ex. $\int \sin^3 x\, dx$ try substitution

let $u = \sin x$

$du = \cos x\, dx$

$\int \sin^3 x\, dx = \int u^3 dx$ but no $\cos x\, dx$

we need trig identities

$\sin^2 x + \cos^2 x = 1$

$\Rightarrow \sin^2 x = 1 - \cos^2 x$

$\int \sin^3 x\, dx = \int \sin^2 x \cdot \sin x\, dx$

$= \int (1 - \cos^2 x) \sin x\, dx$

let $u = \cos x \Rightarrow du = -\sin x\, dx$

$\int (1 - \cos^2 x) \sin x\, dx = -\int (1 - u^2)\, du$

$= -u + \frac{1}{3} u^3 = -\cos x + \frac{1}{3} \cos^3 x + C$

$\blacksquare$

Trig Substitution

$\sqrt{a^2-x^2} \longrightarrow x = a\sin\theta$

$\sqrt{a^2+x^2} \longrightarrow x = a\tan\theta$

$\sqrt{x^2-a^2} \longrightarrow x = a\sec\theta$

note these trig identities

$\sin^2\theta + \cos^2\theta = 1$

$1+\tan^2\theta = \sec^2\theta$

ex. $\displaystyle\int \frac{x^3}{\sqrt{16-x^2}}\,dx$ Let $x = 4\sin\theta$
$dx = 4\cos\theta\, d\theta$

$$= \int \frac{64\sin^3\theta}{\sqrt{16-16\sin^2\theta}}\,dx = \int \frac{64\sin^3\theta}{\sqrt{16(1-\sin^2\theta)}}\,dx$$

$$= \int \frac{64\sin^3\theta}{\sqrt{16\cos^2\theta}}\,dx = \int \frac{64\sin^3\theta}{4\cos\theta}\,4\cos\theta\, d\theta$$

$$= \int 64\sin^3\theta\, d\theta = 64\int (1-\cos^2\theta)\sin\theta\, d\theta$$

let $u = \cos\theta \Rightarrow du = -\sin\theta$

$$= -64\int 1-u^2\, du = -64\left(u - \frac{1}{3}u^3\right)$$

$$= -64\left(\cos\theta - \frac{1}{3}\cos^3\theta\right)$$

Partial Fractions - rewrite $\frac{x}{y}$ as $\frac{A}{B}+\frac{C}{D}$

ex. $\int \frac{x-1}{x^2+3x+2}\,dx = ?$

$$\frac{x-1}{x^2+3x+2} = \frac{x-1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$$x-1 = A(x+2) + B(x+1)$$

$$\Rightarrow x-1 = (A+B)x + (2A+B)$$

$$\Rightarrow A+B=1 \text{ and } 2A+B=-1$$

$$\begin{aligned} A+B &= 1 \\ -2A-B &= 1 \\ \hline -A &= 2 \end{aligned} \qquad A=-2 \Rightarrow B=3$$

$$\int \frac{x-1}{x^2+3x+2}\,dx = \int \frac{-2}{x+1} + \frac{3}{x+2}\,dx$$

$$= \int \frac{-2}{x+1}\,dx + \int \frac{3}{x+2}\,dx$$

$u = x+1$, $du = dx$ $\qquad$ $u_1 = x+2$ $du_1 = dx$

$$= -2\int \frac{1}{u}\,du + 3\int \frac{1}{u_1}\,du_1$$

$$= -2\ln|x+1| + 3\ln|x+2| + C$$

ex $\dfrac{x+1}{(x+3)(x-2)(x-5)} = \dfrac{A}{x+3} + \dfrac{B}{x-2} + \dfrac{C}{x-5}$

if unfactorable quadratic in denominator

then split as $\dfrac{A}{\ } + \dfrac{Bx+C}{\ }$

and use $\displaystyle\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$

ex $\displaystyle\int \frac{x^2-x+6}{x(x^2+3)}\,dx = \int \frac{A}{x} + \frac{Bx+C}{x^2+3}\,dx$

$x^2-x+6 = Ax^2+3A+Bx^2+Cx$

$x^2-x+6 = (A+B)x^2+Cx+3A$ $\quad A=2 \quad C=-1$

$B=-1$

$\displaystyle\int \frac{2}{x}dx - \int \frac{x+1}{x^2+3}\,dx = 2\ln|x| - \int \frac{x}{x^2+3} + \frac{1}{x^2+3}\,dx$

$\displaystyle = 2\ln|x| - \int \frac{x}{x^2+3}\,dx - \int \frac{1}{x^2+3}\,dx$

let $u=x^2+3$ $du=2x\,dx$

$\displaystyle = 2\ln|x| - \frac{1}{2}\int \frac{1}{u}\,du - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{a}}\right)$

$\displaystyle = 2\ln|x| - \frac{1}{2}\ln|x^2+3| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$