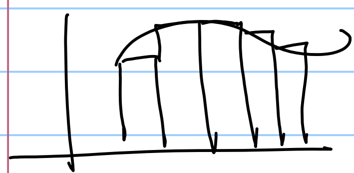
5.9

Note Title 1/9/2008

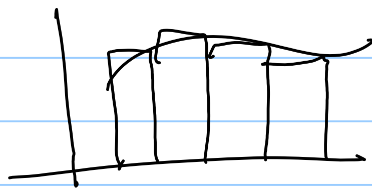
Remember approximating rectangles



left endpoints

$$\int_a^b f(x)\,dx \approx L_n = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

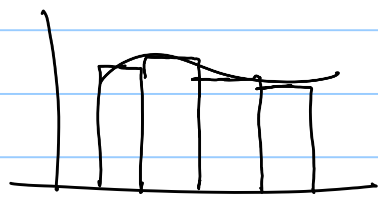
$$= \left[f(x_0)+f(x_1)+f(x_2)+\cdots+f(x_{n-1})\right]\Delta x$$



right endpoints

$$\int_a^b f(x)\,dx \approx R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$= \left[f(x_1)+f(x_2)+\cdots+f(x_n)\right]\Delta x$$



midpoints

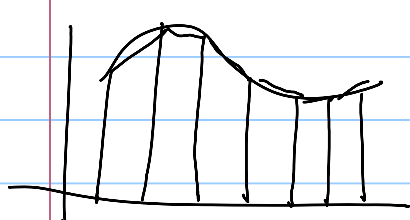
$$\int_a^b f(x)\,dx \approx M_n = \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x$$

$$\bar{x}_i = \tfrac{1}{2}(x_{i-1}+x_i)$$

To get a better approximation, we use trapezoidal rule. idea: average left & right

$$\frac{L_n+R_n}{2} = \frac{\Delta x\left[f(x_0)+f(x_1)+f(x_2)+\cdots+f(x_{n-1})+f(x_1)+f(x_2)+\cdots f(x_n)\right]}{2}$$

$$= \frac{\Delta x}{2}\left[f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\right]$$



Trapezoidal Rule

Average left & right endpoints

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\left[f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1}) + f(x_n)\right]$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

(ex) $\int_1^2 x+1\,dx$ approximate using a) trapezoidal b) midpoint w/ $n=5$

a. $\Delta x = \frac{2-1}{5} = .2$

$x_0 = 1 \quad x_1 = 1.2 \quad x_2 = 1.4 \quad x_3 = 1.6 \quad x_4 = 1.8 \quad x_5 = 2$

$$\int_1^2 x+1\,dx \approx T_5 = \frac{.2}{2}[2+4.4+4.8+5.2+5.6+3]$$

$$= 2.5$$

b. $\Delta x = .2 \quad x_1 = \frac{1+1.2}{2} = 1.1 \quad x_2 = \frac{1.2+1.4}{2} = 1.3$

$x_3 = 1.5 \quad x_4 = 1.7 \quad x_5 = 1.9$

$$\int_1^2 x+1\,dx \approx M_5 = .2[2.1+2.3+2.5+2.7+2.9]$$
$$= 2.5$$

check: $\int_1^2 x+1\,dx = \frac{x^2}{2} + x\Big|_1^2 = 2+2-\frac{1}{2}-1$
$$= 2.5 \checkmark$$

Error Bounds

Suppose $|f''(x)| \le K$ for $a \le x \le b$. If E_T and E_m are the errors in the trapezoidal & Midpoint Rules, then

$$|E_T| \le \frac{K(b-a)^3}{12n^2} \text{ and } |E_m| \le \frac{K(b-a)^3}{24n^2}$$

ex $\int_0^1 e^x\,dx \Rightarrow f(x)=e^x \quad f'(x)=e^x \quad f''(x)=e^x$

$|f''(x)| \le e^1$ for $0 \le x \le 1 \Rightarrow$

$$|E_T| \le \frac{e}{12n^2} \text{ and } |E_m| \le \frac{e}{24n^2}$$

If we wanted to be within .001 of the correct value then

Trapezoidal

$$\frac{e}{12n^2} < .001 \Rightarrow n > \sqrt{\frac{e}{.012}} \approx 226.523$$

So $n = 227$ will ensure accuracy w/in .001

Midpoint

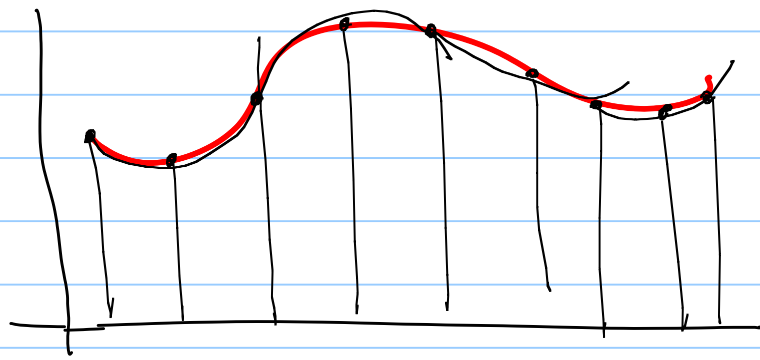
$$\frac{e}{24n^2} < .001 \Rightarrow n > \sqrt{\frac{e}{.024}} \approx 10.64$$

$n = 11$

Simpson's Rule

idea: instead of straight lines (rectangles & trapezoids) use parabolas.

(ex)



Deriving Simpson's Rule

To simplify things let $x_0 = -h,\ x_1 = 0,\ x_2 = h$

The parabola thru P_0, P_1, P_2 has form $y = ax^2 + bx + c$

So area under curve is

$$\int_{-h}^{h} ax^2 + bx + c\,dx = \frac{a}{3}x^3 + \frac{b}{2}x^2 + cx\Big|_{-h}^{h}$$
$$= a\frac{h^3}{3} + b\frac{h^2}{2} + ch - \left(-a\frac{h^3}{3} + b\frac{h^2}{2} - ch\right)$$
$$= 2a\frac{h^3}{3} + 2ch$$
$$= \frac{h}{3}(2ah^2 + 6c)$$

Since Parabola goes thru $P_0(-h, y_0), P_1(0, y_1), P_2(h, y_2)$

$$\Rightarrow \quad y_0 = ah^2 - bh + c$$
$$y_1 = c$$
$$y_2 = ah^2 + bh + c$$

adding $y_0 + 4y_1 + y_2 = ah^2 - bh + c + 4c + ah^2 + bh + c$

$$= 2ah^2 + 6c$$

$$\Rightarrow \int_{-h}^{h} ax^2 + bx + c\,dx = h/3\,(y_0 + 4y_1 + y_2)$$

since shifting the graph horizontally doesn't change the area under the curve

$$\int_{x_2}^{x_4} ax^2 + bx + c\,dx = h/3\,(y_2 + 4y_3 + y_4)$$

$$\vdots$$

$$\int_{x_{n-2}}^{x_n} ax^2 + bx + c\,dx = h/3\,(y_{n-2} + 4y_{n-1} + y_n)$$

$$\int_a^b f(x)\,dx = \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \cdots + \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

$$= \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right]$$

Simpson's Rule

$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

where n is even and $\Delta x = \frac{b-a}{n}$

Error Bound for Simpson's Rule

Suppose $|f^{(4)}(x)| \le K$ for $a \le x \le b$ then

$$|E_S| \le \frac{K(b-a)^5}{180n^4}$$

Approximate using Simpson's

ex. $\int_1^2 \frac{1}{x}\,dx \quad n=4$

$\Delta x = \frac{2-1}{4} = \frac{1}{4} \qquad x_0=1,\ x_1=1.25,\ x_2=1.5,\ x_3=1.75,\ x_4=2$

$\approx S_n = \frac{1/4}{3}\left[\frac{1}{1} + 4\,\frac{1}{5/4} + 2\,\frac{1}{3/2} + 4\cdot\frac{1}{7/4} + \frac{1}{2}\right]$

$= \frac{1}{12}\left(1 + \frac{16}{5} + \frac{4}{3} + \frac{16}{7} + \frac{1}{2}\right) \approx .69325$

$|E_s| \leq \frac{1}{46080} \approx .0000217$

since

$f(x) = x^{-1} \quad f' = -x^{-2} \quad f'' = 2x^{-3} \quad f''' = -6x^{-4}$

$f^{(4)} = 24x^{-5} = 24/x^5 \Rightarrow K=1$