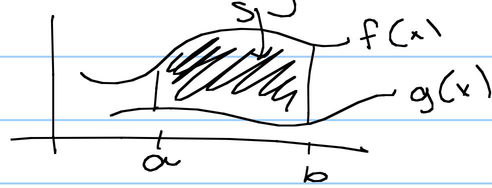
6.1

Note Title 1/14/2008

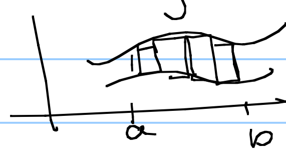
Area Between Curves

Define S as area between two curves $y=f(x)$ and $y=g(x)$ and between $x=a$ and $x=b$ where f & g are continuous on $[a,b]$ and $f \geq g$ $\forall x \in [a,b]$

ex.



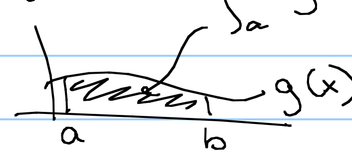
To find area S, we can use approximating rectangles.

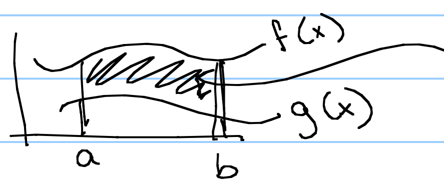


OR

$\int_a^b f(x)dx$

$\int_a^b g(x)$

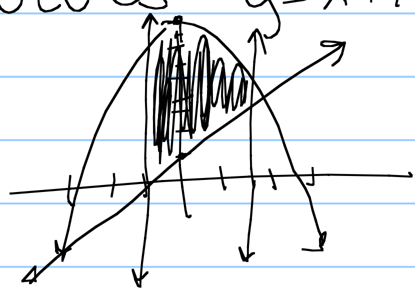




$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$= \int_a^b [f(x) - g(x)]\,dx$$

ex Find the area between the curves $y=x+1$, $y=9-x^2$, $x=-1$, $x=2$



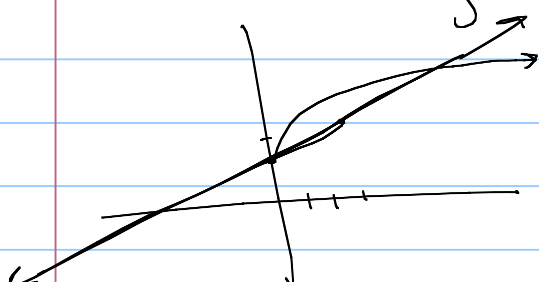
$$\int_{-1}^{2}\left[9-x^2-x-1\right]dx$$

$$=8x-\frac{1}{3}x^3-\frac{1}{2}x^2\Big|_{-1}^{2}$$

$$=-\frac{8}{3}-2+16-\frac{1}{3}+\frac{1}{2}+8$$

$$=39/2=19.5$$

ex find area enclosed by $y=1+\sqrt{x}$ and $y=1+\frac{1}{3}x$



$$1+\sqrt{x}=1+\frac{1}{3}x$$

$$\sqrt{x}=\frac{1}{3}x$$

$$x=\frac{1}{9}x^2$$

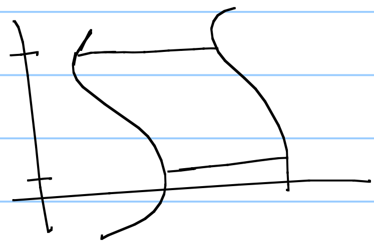
$$0=x\left(\tfrac{1}{9}x-1\right)$$

$$x=0 \text{ or } x=9$$

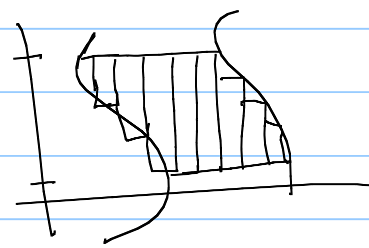
$$\int_0^9\left[(1+\sqrt{x})-(1+\tfrac{1}{3}x)\right]dx=\frac{2}{3}x^{3/2}-\frac{1}{6}x^2\Big|_0^9$$

$$=18-81/6-0=9/2$$

Take the following figure



if we wanted to find enclosed area, our approximating rectangles would be very messy!



instead think of it:

x as a function of y.

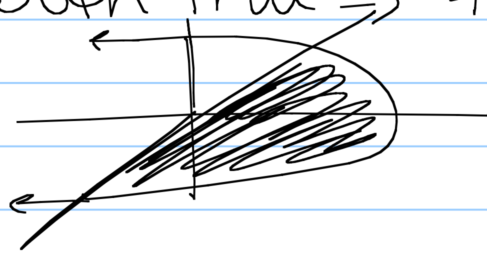
$\leftarrow x_R = $ right x boundary

$$\text{Area} = \int_c^d (x_R - x_L)\,dy$$

$\leftarrow x_L = $ left x boundary

ex. Area between $4x + y^2 = 12$ and $x = y$

graphs intersect when $x = y$ and $4x + y^2 = 12$ are both true $\Rightarrow 4y + y^2 = 12 \Rightarrow y = -6, 2$



$x_R = 3 - \frac{y^2}{4}$ $\qquad$ $x_L = y$

$$\int_{-6}^{2} (x_R - x_L)\,dy = \int_{-6}^{2} 3 - \frac{y^2}{4} - y \; dy$$

$$= 3y - \frac{y^3}{12} - \frac{1}{2}y^2 \Big|_{-6}^{2}$$

$$= 6 - \frac{2}{3} - 2 + 18 - 18 + 18$$

$$= 64/3$$

To know if you should integrate with respect to x or y

- Can you solve the eqns for x or y
  - if $y = \_\_$ integrate w/respect to x
  - if $x = \_\_$ " " y
- Is one graph always on top/right of other
  - always on top - w/respect to x
  - always on right - w/respect to y

Areas enclosed by Parametric Curves

We know Area under $y=F(x)$ between a & b
is $A=\int_a^b F(x)\,dx=\int_a^b y\,dx$

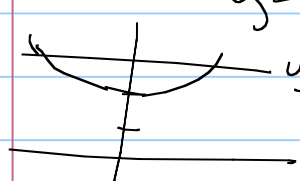
if we also know $x=f(t)$ and $y=g(t)$
for $\alpha<t<\beta$ then

$$\frac{dx}{dt}=f'(t)\Rightarrow dx=f'(t)\,dt$$

and so

$$A=\int_a^b y\,dx=\int_\alpha^\beta g(t)f'(t)\,dt \quad \left[\text{or } \int_\beta^\alpha g(t)f'(t)\,dt\right.$$

ex #32
Find area enclosed between $x=t-1/t$, $y=t+1/t$ and $y=2.5$



$$2.5=\frac{t^2+1}{t}$$
$$2.5t=t^2+1$$
$$t^2-2.5t+1=0$$
$$(2t-1)(t-1)=0$$
$$t=1/2,2 \Rightarrow \alpha=1/2\ \ \beta=2$$

$$A=\int_{t=1/2}^{t=2} 2.5-y\,dx$$

$$\frac{dx}{dt}=1+1/t^2$$
$$dx=(1+1/t^2)\,dt$$

$$A = \int_{t=1/2}^{t=2} 2.5 - t - 1/t \, dx$$

$$= \int_{t=1/2}^{t=2} (2.5 - t - t^{-1})(1 + t^{-2}) \, dt$$

$$= \int_{t=1/2}^{t=2} 2.5 + 2.5t^{-2} - t - t^{-1} - t^{-1} - t^{-3} \, dt$$

$$= \int_{t=1/2}^{t=2} 2.5t^{-2} + 2.5 - t - 2t^{-1} - t^{-3} \, dt$$

$$= -\frac{5}{2} t^{-1} + \frac{5}{2} t - \frac{1}{2} t^2 - 2\ln|t| + \frac{1}{2} t^{-2} \Big|_{t=1/2}^{t=2}$$

$$= -\frac{5}{2} \cdot \frac{1}{2} + \frac{5}{2} \cdot 2 - 2 - 2\ln 2 + \frac{1}{2} \cdot \frac{1}{4} - \Big(-\frac{5}{2} \cdot 2 + \frac{5}{2} \cdot 1/2$$
$$- 1/8 - 2\ln(1/2) + 1/2 \cdot 4 \Big)$$

$$= -\frac{5}{4} + 5 - 2 - 2\ln 2 + 1/8 + 5 - 5/4 + 1/8 + 2\ln(1/2) - 2$$

$$= 1 + 5 - 2 + 5 - 5/4 - 2 - 2\ln 2 + 2\ln 1/2$$

$$= \frac{15}{4} - 2(\ln 2 - \ln 1/2)$$

$$= \frac{15}{4} - 2\ln 4$$