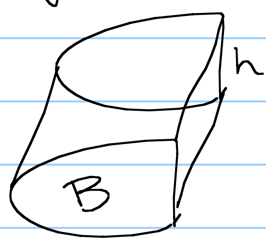
6.2

Note Title 1/21/2008

Volume - Disks Method

ex) Volume of right cylinder



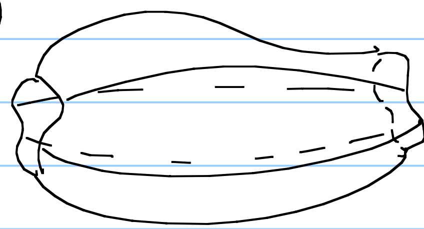
$V = B \cdot h$

where B = area base

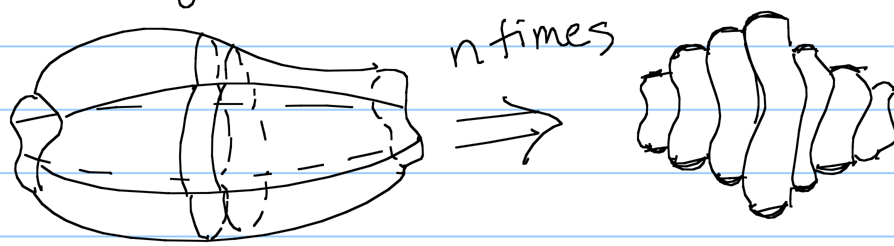
h = height

But what if shape is NOT a right cylinder?

ex.)



idea: take n cross-sections and approximate the volume as a right cylinder



See pg. 449 for better drawing!

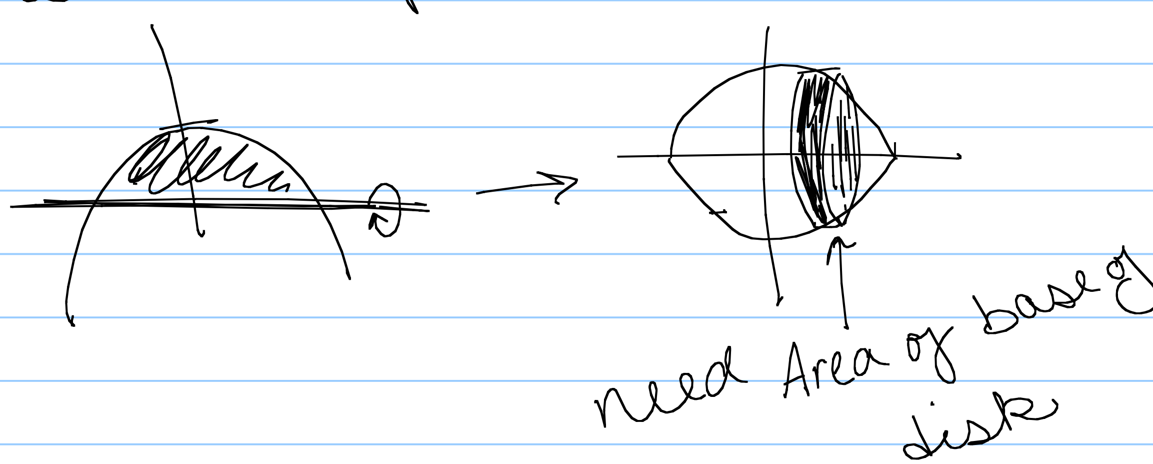
As $n \to \infty$ approximation will get better & better.

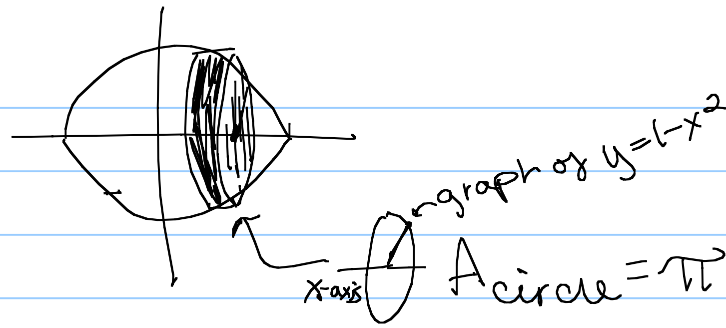
Let S be a solid that lies between $x=a$ and $x=b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the volume of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i)\Delta x = \int_a^b A(x)\,dx$$

ex) find volume of figure obtain by rotating the curve $y = 1-x^2$ about the x-axis bounded by $y=0$

first draw pic





$$A_{circle} = \pi r^2$$

$$= \pi(1-x^2)^2$$

$$V(x) = \int_{-1}^{1} \pi(1-x^2)^2\,dx$$

$$= \pi\int_{-1}^{1} 1-2x^2+x^4\,dx$$

$$= \pi\left[x-\frac{2}{3}x^3+\frac{1}{5}x^5\right]_{-1}^{1}$$

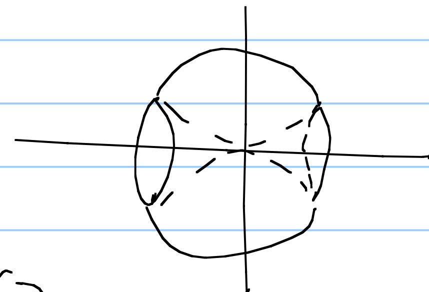
$$= \pi\left[\left[1-\frac{2}{3}+\frac{1}{5}\right]-\left[-1+\frac{2}{3}-\frac{1}{5}\right]\right]$$

$$= \pi\left[2-\frac{4}{3}+\frac{2}{5}\right]$$

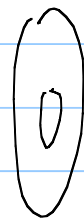
$$= \frac{16\pi}{15}$$

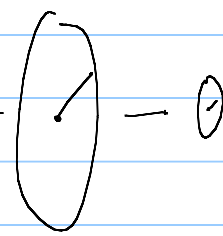
try one. #6

$y = \frac{1}{4}x^2 \quad y = 5 - x^2$ about x-axis



now it's not a disk, it's a <u>washer</u>



Area = 

$$\text{Area}_{out} = \pi(5-x^2)^2 \qquad \text{Area}_{in} = \pi(\tfrac{1}{4}x^2)^2$$

$$A(x) = \pi(5-x^2)^2 - \pi(\tfrac{1}{4}x^2)^2$$

$$= \pi\left(25 - 10x^2 + x^4 - \tfrac{1}{16}x^4\right)$$

$$V(x) = \pi\int_{-2}^{2} 25 - 10x^2 + \tfrac{15}{16}x^4\,dx$$

$$= \pi\left[25x - \tfrac{10}{3}x^3 + \tfrac{3}{16}x^5\right]_{-2}^{2}$$

$$= \pi\left[50 - \tfrac{80}{3} + 6 - \left(-50 - -\tfrac{80}{3} - 6\right)\right]$$

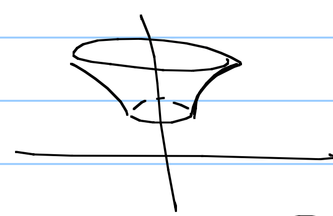
$$= \pi\left[50 - \frac{80}{3} + 6 + 50 - \frac{80}{3} + 6\right]$$

$$= \pi\left(100 - \frac{160}{3} + 12\right)$$

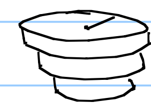
$$= \frac{176\pi}{3}$$

Rotating about y-axis

ex. $y = \ln x,\ y = 1,\ y = 2,\ x = 0$ about y-axis



now disks



$y = \ln x$

$e^{y} = x$

So to find $A(y) = \pi(e^{y})^2$

$$V = \pi\int_1^2 e^{2y}\,dy = \frac{1}{2}\pi\int_{y=1}^{y=2} e^{u}\,du = \frac{1}{2}\pi e^{2y}\Big|_1^2$$

let $u = 2y$ $\quad du = 2dy$

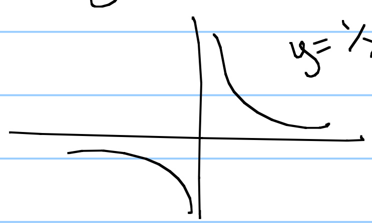
$\frac{1}{2}du = dy$

$$= \frac{1}{2}\pi e^4 - \frac{1}{2}\pi e^2$$

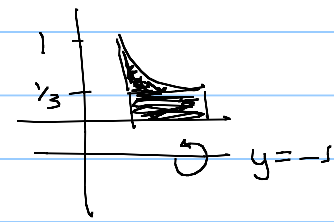
Rotating about lines that are not x- or y-axis

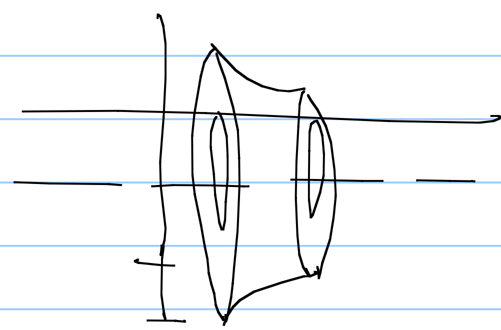
ex. #10

$y = 1/x$, $y = 0$, $x = 1$, $x = 3$, about $y = -1$

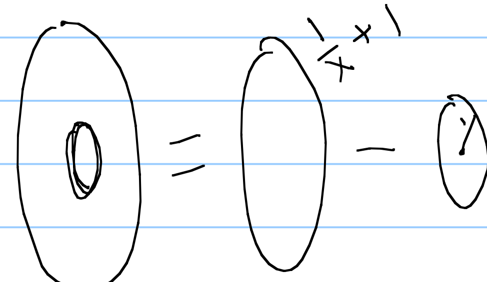


add restrictions





washer



$$A(x) = \pi\left(\frac{1}{x}+1\right)^2 - \pi(1)^2$$

$$V = \pi\int_1^3 \frac{1}{x^2} + \frac{2}{x} + 1 - 1 \, dx$$

$$= \pi\int_1^3 x^{-2} + 2x^{-1}\,dx = \pi\left[-\frac{1}{x} + 2\ln x\right]_1^3$$

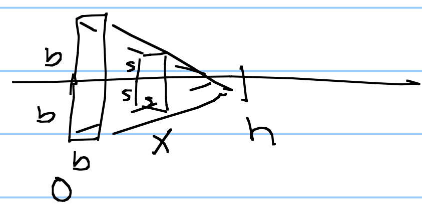
$$= \pi\left(2/3 + 2\ln 3\right)$$

Doesn't have to be a <u>solid of revolution</u>

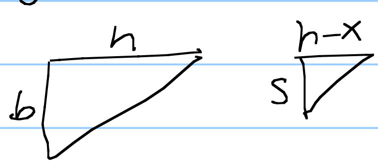
Remember $V=\int_a^b A(x)\,dx$ where

$A(x)$ = cross-sectional area

ex. Find the volume of a pyramid w/ height h and base w/ dimensions $b \times 2b$



by similiar △'s



$$\frac{s}{h-x}=\frac{b}{h}$$

$$\Rightarrow s=\frac{b}{h}(h-x)$$

$$A(x)=2s*s$$
$$=2s^2$$
$$=2\left(\frac{b}{h}(h-x)\right)^2$$
$$=\frac{2b^2}{h^2}(h-x)^2$$

$$A(x)=\frac{2b^2}{h^2}(h^2-2xh+x^2)$$

$$V = \int_0^h A(x)\,dx$$

$$= \int_0^h \frac{2b^2}{h^2}\left(h^2 - 2xh + x^2\right)dx$$

$$= \frac{2b^2}{h^2}\int_0^h h^2 - 2hx + x^2\,dx$$

$$= \frac{2b^2}{h^2}\left[h^2x - hx^2 + \frac{1}{3}x^3\right]_0^h$$

$$= \frac{2b^2}{h^2}\left[h^3 - h^3 + \frac{1}{3}h^3 - 0\right]$$

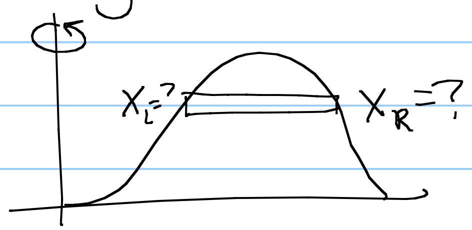
$$= \frac{2}{3}b^2h$$

$\blacksquare$

Cylindrical Shells

Not all problems are 'easily' solved by the above methods.

ex $y = 2x^2 - x^3$

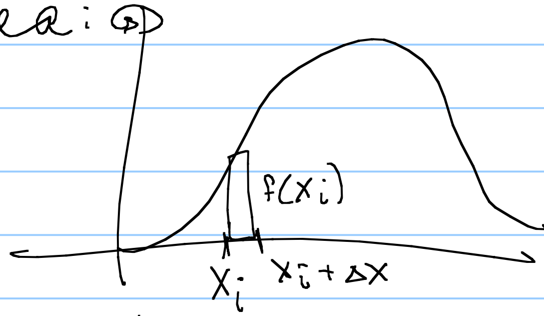


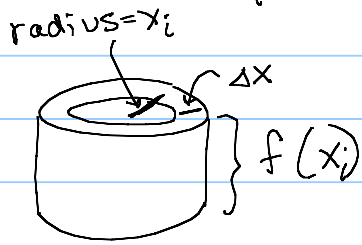
$x_R = x_L =$ "$y = 2x^2 - x^3$ solved for x"

problem!

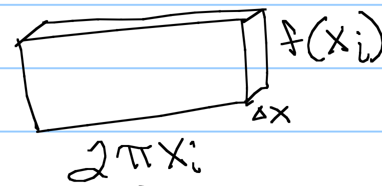
So instead of using disks method, we use shells method.

idea:





flatten out shell



circumference of a circle is $2\pi r$.

$$\Rightarrow V_{shell} = 2\pi x_i f(x_i)\Delta x$$

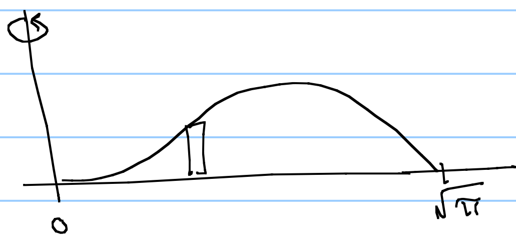
Take A LOT of shells (let $n\to\infty$)

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x_i f(x_i)\Delta x$$

$$\Rightarrow V = \int_a^b 2\pi x f(x)\,dx \quad \text{about } y\text{-axis}$$

$$V = \int_a^b 2\pi y f(y)\,dy \quad \text{about } x\text{-axis}$$

ex. $y = \sin(x^2)$ from 0 to $\sqrt{\pi}$ about y-axis
find volume using shells.



$$V = \int_0^{\sqrt{\pi}} 2\pi x \sin(x^2)\,dx$$

$$= \pi \int_0^{\sqrt{\pi}} \sin(x^2) 2x\,dx \quad \text{let } u = x^2 \quad du = 2x\,dx$$

$$= \pi \int_{x=0}^{x=\sqrt{\pi}} \sin u\,du$$

$$= -\pi\left[\cos x^2\right]_0^{\sqrt{x}} = -\pi(-1-1) = 2\pi$$