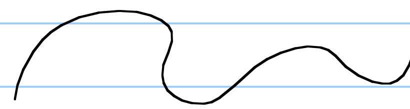
6.3

Note Title 1/28/2008

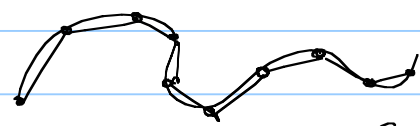
Arc Length

idea:



lay a string on the curve, then stretch the string along a ruler, this is arc length.

how we find it:



take n intervals & find the length of each line segment & sum them.

As $n \to \infty$ the approximation gets better & better

The length of line segments can be found using distance formula.

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

Assume $x = f(t) \Rightarrow f'(t) \approx \frac{\Delta x}{\Delta t} \Rightarrow \Delta x = f'(t)\Delta t$

$y = g(t) \Rightarrow g'(t) \approx \frac{\Delta y}{\Delta t} \Rightarrow \Delta y = g'(t)\Delta t$

$$\Rightarrow d = \sqrt{(f'(t)\Delta t)^2 + (g'(t)\Delta t)^2}$$

$$d = \sqrt{\Delta t^2 \left(f'(t)^2 + g'(t)^2 \right)} = \sqrt{f'(t)^2 + g'(t)^2}\, \Delta t$$

if we add all the lengths together

$$L = \sum_{i=1}^{n} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, \Delta t$$

if we let $n \to \infty$ we get

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

<u>Arc Length formula</u>

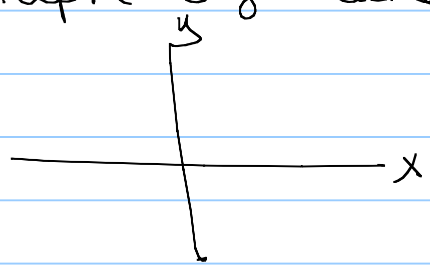
If a smooth curve with parametric equations $x = f(t)$, $y = g(t)$, $a \le t \le b$, is traversed once as t increases from a to b then it's length is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

ex: graph & find length of curve

$x = e^t + e^{-t}, \quad y = 5 - 2t, \quad 0 \le t \le 3$

graph by hand



t	x	y
0	2	5
1		3
2		1
3		-1

on calculator

$\boxed{\text{mode}}$

$\frac{dx}{dt} = e^t - e^{-t}$ $\qquad$ $\frac{dy}{dt} = -2$

Length

$$L = \int_0^3 \sqrt{(e^t - e^{-t})^2 + (-2)^2}\, dt$$

$$= \int_0^3 \sqrt{e^{2t} - 2 + e^{-2t} + 4}\ dt$$

$$= \int_0^3 \sqrt{e^{2t} + 2 + e^{-2t}}\ dt$$

$$= \int_0^3 \sqrt{(e^t + e^{-t})^2}\ dt$$

$$= \int_0^3 e^t + e^{-t}\ dt$$

$$= e^t - e^{-t}\Big|_0^3 = e^3 - e^{-3} - 1 + 1$$

$$= e^3 - e^{-3}$$

If $y = f(x)$, $a \leq x \leq b$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

If $x = f(y)$, $a \leq y \leq b$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\ dy$$

ex) $y = \frac{x^2}{2} - \frac{\ln x}{4}$, $2 \le x \le 4$

$$L = \int_2^4 \sqrt{1 + \left(x - \frac{1}{4x}\right)^2}\, dx \qquad \frac{dy}{dx} = x - \frac{1}{4x}$$

$$= \int_2^4 \sqrt{1 + x^2 - \frac{1}{2} + \frac{1}{16x^2}}\, dx$$

$$= \int_2^4 \sqrt{x^2 + \frac{1}{2} + \frac{1}{16x^2}}\, dx$$

$$= \int_2^4 \sqrt{\left(x + \frac{1}{4x}\right)^2}\, dx$$

$$= \int_2^4 x + \frac{1}{4x}\, dx$$

$$= \frac{1}{2}x^2 + \frac{1}{4}\ln x \Big|_2^4 = 8 + \frac{1}{4}\ln 4 - 2 - \frac{1}{4}\ln 2$$

$$= 6 + \frac{1}{4}\ln 4 - \frac{1}{4}\ln 2$$

or $6 + \frac{1}{4}\ln 2$