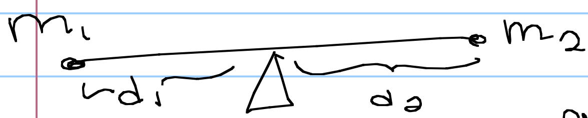
6.5

Note Title 2/1/2008

Moments & Centers of mass

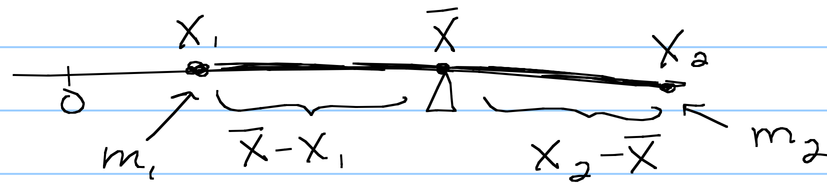


Assume two masses are attached to a rod & the rod is placed on fulcrum so that rod balances.

Let d_1 & d_2 be distances then

$$m_1 d_1 = m_2 d_2$$

Place this on x-axis & label stuff



So now,

$$m_1(\bar{x}-x_1) = m_2(x_2-\bar{x})$$

$$\Rightarrow m_1\bar{x} + m_2\bar{x} = m_1x_1 + m_2x_2$$

$$\Rightarrow \bar{x} = \frac{m_1x_1 + m_2x_2}{m_1+m_2}$$

The #s m_1x_1 and m_2x_2 are called <u>moments</u>. $\bar{x}$ is center of mass

If we have a bunch of masses

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_i x_i}{m}$$

We call $M = \sum_{i=1}^{n} m_i x_i$ the moment of the system about the origin.

Two-dimensional case

If instead of being on x-axis our points are (x, y) and

Moment of the system about the y-axis

$$M_y = \sum_{i=1}^{n} m_i x_i$$

measures tendency to rotate about y-axis

Moment of the system about the x-axis

$$M_x = \sum_{i=1}^{n} m_i y_i$$

measures tendency to rotate about x-axis

The center of mass, $(\bar{x}, \bar{y})$ is

$$\bar{x} = \frac{M_y}{m} \qquad \bar{y} = \frac{M_x}{m}$$

ex) $m_1 = 6$ $(1, -2)$ $\quad m_2 = 5$ $(3, 4)$ $\quad m_3 = 1$ $(-3, -7)$ $\quad m_4 = 4$ $(6, -1)$

find center of mass

$$\bar{x} = \frac{6\cdot 1 + 5\cdot 3 + -3\cdot 1 + 6\cdot 4}{16} = 42/16$$

$$\bar{y} = \frac{-2\cdot 6 + 5\cdot 4 - 7\cdot 1 - 1\cdot 4}{16} = -3/16$$

$(42/16, -3/16)$

Now consider a funny shaped flat plate (called a lamina) with uniform density ρ

Centroid - the center of mass of the plate.

$$M_y = \rho \int_a^b x f(x)\, dx$$

$$M_x = \rho \int_a^b \frac{1}{2}[f(x)]^2\, dx$$

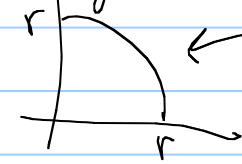
the centroid is located at

$$\bar{x} = \frac{1}{A}\int_a^b x f(x)\, dx \qquad \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2\, dx$$

where A = area of plate.

(ex) find center of mass

$\rho = 2$



← quarter circle.

$x^2 + y^2 = r^2$

$y = \sqrt{r^2 - x^2}$

$$\bar{x} = \frac{1}{\pi r^2/4} \int_0^r x\sqrt{r^2 - x^2}\,dx = \frac{4}{\pi r^2}\int_0^r -\frac{1}{2}\sqrt{u}\,du$$

Let $u = r^2 - x^2$ $\quad -\frac{1}{2}du = -\cancel{2}x\,dx$

$$= \frac{-2}{\pi r^2}\left[\frac{2}{3}u^{3/2}\right]_0^r = -\frac{4}{3\pi r^2}\left[(r^2 - x^2)^{3/2}\right]_0^r$$

$$= -\frac{4}{3\pi r^2}\left[(r^2 - r^2)^{3/2} - (r^2 - 0)^{3/2}\right]$$

$$= -\frac{4}{3\pi r}$$

$$\bar{y} = \frac{1}{\pi r^2/4}\int_0^r \frac{1}{2}\left(\sqrt{r^2 - x^2}\right)^2 dx$$

$$= \frac{2}{\pi r^2}\int_0^r r^2 - x^2\,dx = \frac{2}{\pi r^2}\left(r^2 x - \frac{1}{3}x^3\right|_0^r$$

$$= \frac{2}{\pi r^2}\left(r^3 - \frac{1}{3}r^3 - 0\right) = \frac{4r}{3\pi}$$

$$\left(-\frac{4}{3\pi r}, \frac{4r}{3\pi}\right)$$