7.4

Note Title 3/3/2008

Law of Natural Growth (if $K>0$) or Decay (if $K<0$)

$$\frac{dy}{dt} = K \cdot y$$

Says the rate of growth is proportional to population size

i.e. double the population ⇒ double growth rate

This is seperable as

$$\int \frac{1}{y}\,dy = \int K\,dt$$

$$\ln|y| = Kt + C$$

$$|y| = e^{Kt+C}$$

$$y = e^{Kt}e^{C} \quad \text{let } e^{C} = A$$

$$y = Ae^{Kt} \quad \text{where } A \text{ is arbitrary constant.}$$

notice if $t=0$ then $y=A$ so A is initial value.

This leads to a very important finding:

The Solution to

$$\frac{dy}{dt} = Ky, \quad y(0) = y_0$$

is

$$y(t) = y_0 e^{Kt}$$

Significance of K

$$\frac{dP}{dt} = KP \Rightarrow \frac{\frac{dP}{dt}}{P} = K$$

So K is growth rate divided by Population size, this is called relative growth rate.

ex. A population of bacteria is growing at $\frac{dP}{dt} = 10$ bacteria/min when $t=5$. If population at $t=5$ is 50 bact. then relative growth rate is
$10/50 = 20\%$ growth per min

Saying "growth rate is constant" says
$$\frac{dP}{dt} = Kt$$

Radioactive Decay

$\frac{dm}{dt} = Km$ where m = remaining mass

We know a soln to this diff. eq.
$$m(t) = m_0 e^{Kt}$$

half-life - amount of time required for half of a quantity to decay.

ex. Bismuth-210 has a half-life of 5 days. You have an initial sample of 800 mg. Find $m(t)$?

$$\frac{dm}{dt} = K \cdot m \longrightarrow m(t) = 800e^{Kt}$$

$$m(5) = 400$$

$$800e^{K \cdot 5} = 400$$

$$K = \frac{\ln \frac{1}{2}}{5}$$

$$m(t) = 800e^{\frac{\ln \frac{1}{2}}{5}t} \xrightarrow{\text{or}} 800 \cdot 2^{-t/5}$$

Newton's Law of Cooling

$\frac{dT}{dt} = K(T - T_s)$ where T_s = temp of surrounding area.

t = time T = temp of object

Says rate that an object cools or warms is proportional to the difference of temps between the object & the surroundings

If we let $y(t) = T(t) - T_s$ we get

$$\frac{dy}{dt} = K \cdot y \Rightarrow y(t) = y_0 e^{Kt} \Rightarrow T(t) = y_0 e^{Kt} + T_s$$

(ex) A thermometer is taken from a 20°C room outside where it is 5°C. After one min. it reads 12°C. Find $T(t)$?

$T_0 = 20°$ $\quad T_s = 5°$ $\quad T(1) = 12°$

$$\frac{dT}{dt} = k(T-5)$$

let $y(t) = T - 5 \Rightarrow y_0 = 15$ and $y(1) = 7$

and $T(t) = y(t) + 5$

$$\frac{dy}{dt} = k \cdot y \Rightarrow y(t) = y_0 e^{kt}$$
$$= 15e^{kt}$$

$$y(1) = 15e^{k} = 7$$
$$k = \ln 7/15$$

$$y(t) = 15e^{\ln 7/15\, t}$$
$$= 15(7/15)^t$$

$$T(t) = 15(7/15)^t + 5$$