7.5

Note Title 3/3/2008

Population growth $\frac{dP}{dt} = k \cdot P$ in ideal conditions. But what about not ideal conditions (more realistic)

When P is small $\frac{dP}{dt} = k \cdot P$ since there's lots of food, water, space, etc.

note: k vs K

But if P gets too big (>K called <u>carrying capacity</u>) it will start to decrease.

We express this as

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$$

when $P \to 0$ then $kP\left(1 - \frac{P}{K}\right) \to kP$

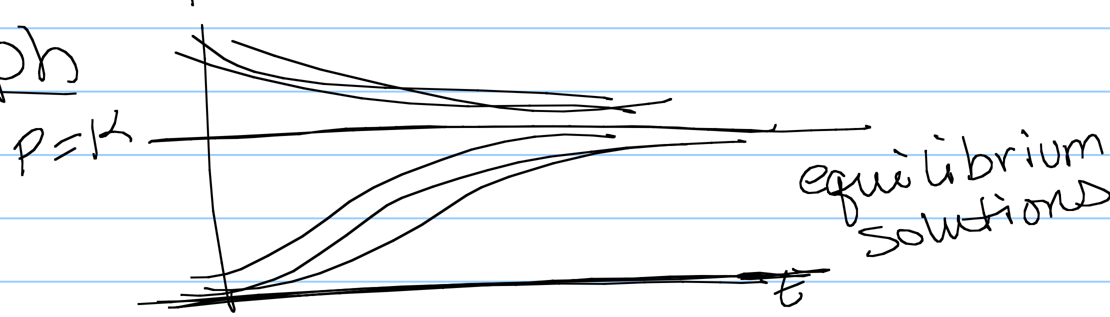
if $P > K$ then $\frac{P}{K} > 1 \Rightarrow kP\left(1 - P/K\right) < 0$

if $P < K$ then $P/K < 1 \Rightarrow kP\left(1 - P/K\right) > 0$

<u>Logistic Diff. Eq</u>

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$$

<u>Graph</u>



Solving Logistic Diff. Eq

$\frac{dP}{dt} = kP\left(1-\frac{P}{K}\right)$ is seperable

$$\int \frac{1}{P(1-\frac{P}{K})}\,dP = \int k\,dt$$

$$\int \frac{1}{\frac{PK}{K}-\frac{P^2}{K}}\,dP = kt + C$$

$$\int \frac{K}{PK-P^2}\,dP = kt + C$$

$$\int \frac{K}{P(K-P)}\,dP = kt + C$$

use Partial fractions

$$\frac{A}{P} + \frac{B}{K-P} = \frac{K}{P(K-P)} \Longrightarrow A(K-P) + BP = K$$

$$AK - AP + BP = K$$

$$P(B-A) + AK = K$$

$$B - A = 0 \qquad A = 1$$

$$B = 1$$

$$\int \frac{1}{P} + \frac{1}{K-P}\,dP = kt + C$$

$$\ln|P| - \ln|K-P| = kt + C$$

$$\ln|K-P| - \ln|P| = -kt - C$$

$$\ln\left|\frac{K-P}{P}\right| = -kt - C$$

$$\left|\frac{K-P}{P}\right| = e^{-kt-C} = e^{-C}e^{-kt} \quad \text{let } A = \pm e^{-C}$$

$$\frac{K-P}{P} = Ae^{-kt}$$

Take a break to find A

let $t=0$ $P=P_0$

$$\frac{K-P_0}{P_0} = Ae^{-k \cdot 0} \Rightarrow \frac{K-P_0}{P_0} = A$$

Substitute in

$$\frac{K-P}{P} = \frac{K-P_0}{P_0} e^{-kt}$$

Solve this for P

$$\frac{K}{P} - 1 = \frac{K-P_0}{P_0} e^{-kt}$$

$$P(t) = \frac{K}{1+Ae^{-kt}} \quad \text{where } A = \frac{K-P_0}{P_0}$$

So, the soln to Logistic Diff. Eq.

$$\frac{dP}{dt} = kP\left(1-\frac{P}{K}\right)$$

is $P(t) = \dfrac{K}{1+Ae^{-kt}}$ where $A = \dfrac{K-P_0}{P_0}$

ex. The total # of people infected w/ a virus often grows like a logistic curve. Suppose that 10 people originally have the virus & early on the # of people infected increases with $k = 1.78$. It is estimated that 5000 people will become infected, in the long run.

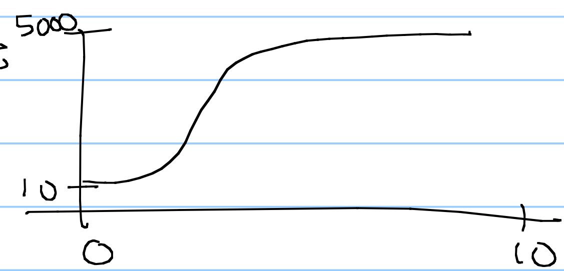
a.) find a logistic model?

$k = 1.78 \quad K = 5000 \quad P_0 = 10$

$$A = \frac{5000-10}{10} = 499$$

$$P(t) = \frac{5000}{1+499e^{-1.78t}}$$

b.) graph it:



c.) Use graph to estimate length of time until the rate at which people are being infected starts to decrease? How many people are infect then?

$t \approx 3.5$ wks.

$P \approx 2500$ people