8.1

Note Title 3/17/2008

Sequence - a list of numbers in a definite order

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

other notations $\{a_1, a_2, \ldots\}$, $\{a_n\}$, $\{a_n\}_{n=1}^{\infty}$

Some sequences have formulas

ex. $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ or $a_n = \frac{n}{n+1}$ or $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$

$\left\{\frac{(-1)^n}{2^n}\right\}$ or $a_n = \frac{(-1)^n}{2^n}$ or $\left\{\frac{-1}{2}, \frac{1}{4}, \frac{-1}{8}, \frac{1}{16}, \ldots\right\}$

by default $n=1$

ex. find a formula for $\left\{\frac{2}{2}, -\frac{4}{3}, \frac{6}{4}, -\frac{8}{5}, \frac{10}{6}, \ldots\right\}$

signs $+, -, +, -, + \ldots$ $(-1)^{n+1}$ (or $(-1)^{n-1}$)

numerators $2, 4, 6, 8, 10, \ldots$ $2n$

denominators $2, 3, 4, 5, 6, \ldots$ $n+1$

So $a_n = \frac{(-1)^{n+1} 2n}{n+1}$

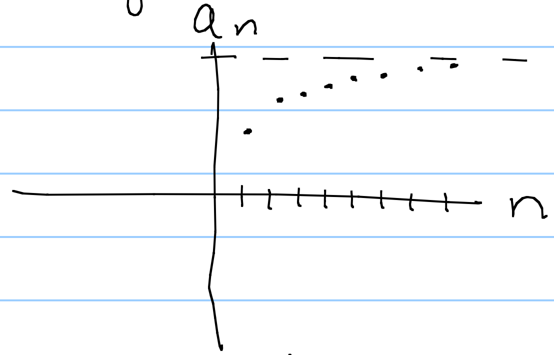
Not all sequences have nice formulas.

ex. Fibonacci Sequence

$F_1 = 1$, $F_2 = 1$ $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$

$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

ex. Let's graph $a_n = \frac{n}{n+1} = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{7}{8}, \frac{8}{9}, \ldots$



notice when $n = 1000$
$a_n = \frac{1000}{1001} \approx .99900$

We say $\lim_{n\to\infty} a_n = 1$

If $\lim_{n\to\infty} a_n$ exists we say the sequence <u>converges</u>. Otherwise it <u>diverges</u>

We have same limit properties

If $\{a_n\}$ & $\{b_n\}$ are convergent & c is constant

$\lim_{n\to\infty} a_n + b_n = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$ (dido for $-, \times, \div$)

$\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n$

$\lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p$ if $p > 0$ & $a_n > 0$

Squeeze Thm

If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$
then $\lim_{n\to\infty} b_n = L$

If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$

ex. find $\lim_{n\to\infty} \frac{n+1}{3n-1} = \lim_{n\to\infty} \frac{1+1/n}{3-1/n} \to \frac{1+0}{3-0} = 1/3$

Reminder, L'Hospitals

If $\lim_{x\to a} \frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

ex. find $\lim_{n\to\infty} \frac{\sqrt{n}}{1+\sqrt{n}} \to \frac{\infty}{\infty} \xrightarrow{H} \lim_{n\to\infty} \frac{\frac{1}{2}n^{-1/2}}{\frac{1}{2}n^{-1/2}} = \lim_{n\to\infty} 1 = 1$

ex. $\lim_{n\to\infty} (-1)^n$ dne since $= -1, 1, -1, 1, -1, 1, -1, \ldots$

Sequence r^n

r^n is convergent if $-1 < r \leq 1$

r^n is divergent for all other r's.

In/Decreasing

A sequence is increasing if $a_n < a_{n+1}$

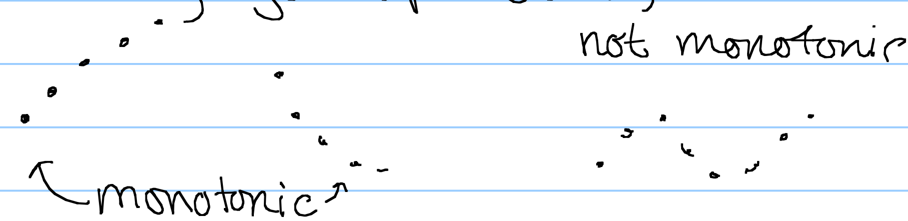
(aka $a_1 < a_2 < a_3 < \cdots$)

and decreasing if $a_n > a_{n+1}$

(aka $a_1 > a_2 > a_3 > \cdots$)

and monotonic if it's either increasing or decreasing.

(strictly goes up or down, doesn't bounce)



(ex.) Is $\frac{2n}{n+1}$ in/decreasing?

$$\frac{2n}{n+1} \overset{?}{<} \frac{2(n+1)}{(n+1)+1} = \frac{2n+2}{n+2}$$

$$\frac{2n}{n+1} \overset{?}{<} \frac{2n+2}{n+2}$$

$$2n(n+2) \overset{?}{<} (n+1)(2n+2)$$

$$2n^2+4n \overset{?}{<} 2n^2+4n+2$$

yes $2n^2+4n < 2n^2+4n+2$

$$\Rightarrow \frac{2n}{n+1} < \frac{2(n+1)}{(n+1)+1} \Rightarrow a_n < a_{n+1}$$

$$\Rightarrow \frac{2n}{n+1} \text{ is increasing}$$

$\blacksquare$

<u>Bounded Sequences</u>

a_n is <u>bounded above</u> if $\exists$ a number M:

$$a_n \leq M$$

it is <u>bounded below</u> if $\exists$ a # m:

$$a_n \geq m$$

if a_n is bounded above and below it is a <u>bounded sequence</u>.

ex. $\{n\}_{n=0}^{\infty}$ is bounded below since

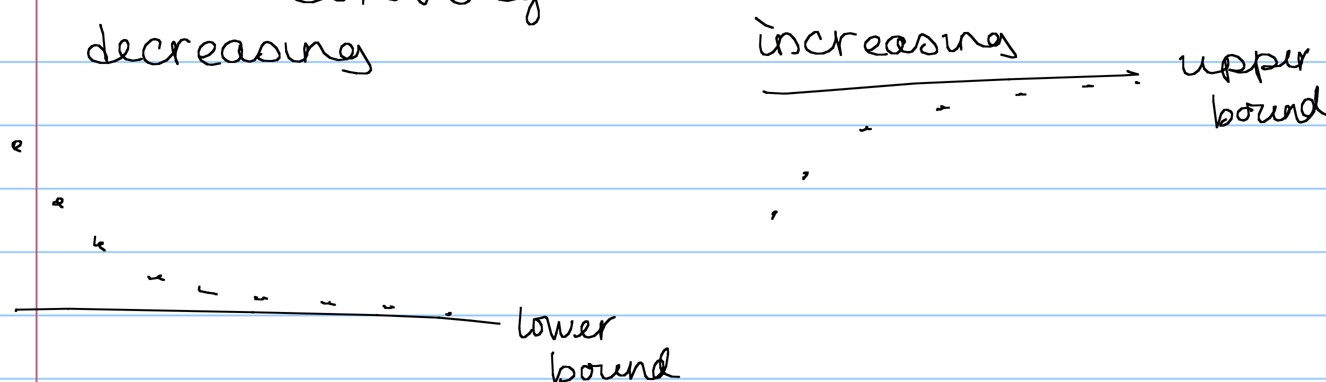
$$a_n \geq 0$$

but not bounded above.

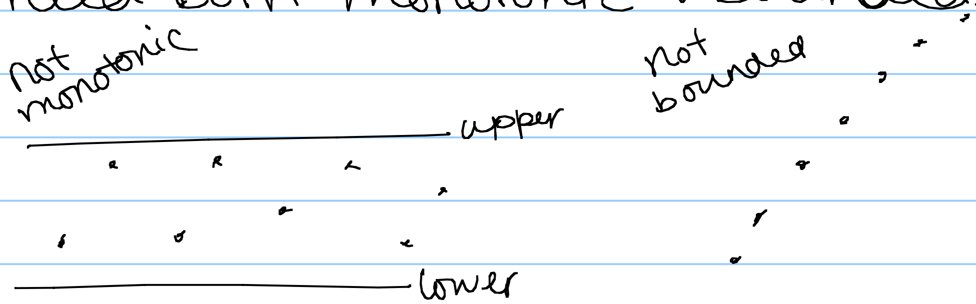
ex. $\{\frac{1}{n}\}_{n=1}^{\infty}$ is a bounded sequence since $0 \leq \frac{1}{n} \leq 1$

<u>Monotonic Sequence Thm</u>

Every bounded, monotonic sequence converges.



need both monotonic & bounded.



ex. $\{\frac{1}{n}\}_{n=1}^{\infty}$ converges since it is bounded (see ↑) and it is decreasing $\left(\frac{1}{n} > \frac{1}{n+1}\right)$