8.3

Note Title 3/17/2008

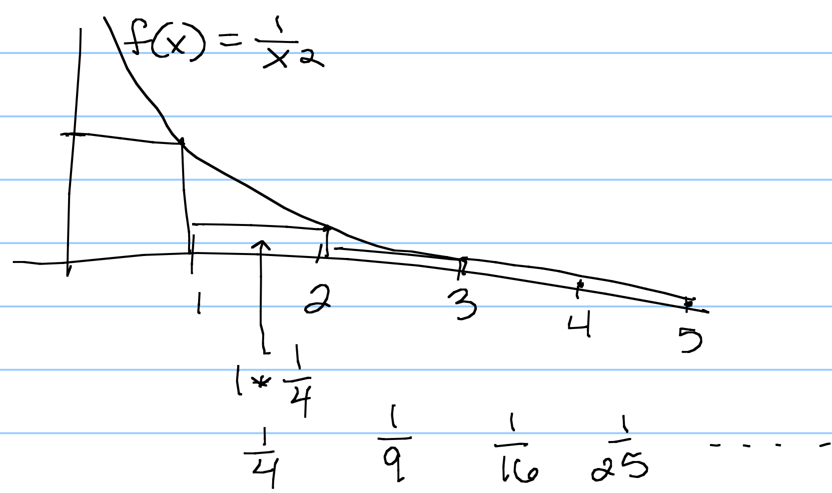
In this section assume terms are all positive so partial sums are strictly increasing.

The Integral Test

If $\int_1^\infty f(x)\,dx$ is convergent, then $\sum a_n$ is convergent

If $\int_1^\infty f(x)\,dx$ is divergent, then $\sum a_n$ is divergent.

ex. $\sum \frac{1}{n^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots$



So if $\int_1^\infty f(x)\,dx = S$ then $\sum a_n = S$

since $\int_1^\infty f(x)\,dx = \sum 1 * a_n$

(same if $\int_1^\infty f(x)\,dx = \infty$ or dne)

note: doesn't say what $\sum a_n$ converges to! only says it converges.

ex. does $\sum \frac{n}{n^2+1}$ converge or diverge?

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_1^{\infty} \frac{1}{u}\,du$$

let $u = x^2+1$

$\frac{1}{2}du = \cancel{2}x\,dx$

$$= \frac{1}{2}\ln(x^2+1)\Big|_1^{\infty}$$

$$= \frac{1}{2}\lim_{a\to\infty} \ln(a^2+1) - \frac{1}{2}\ln 2$$

$$= \infty$$

diverges!

ex. $\sum \frac{1}{n^2+1}$ converge? diverge?

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \tan^{-1}(x)\Big|_1^{\infty} = \lim_{a\to\infty} \tan^{-1}(a) - \tan^{-1} 1$$

$$= \pi/2 - \pi/4 = \pi/4$$

$\int_1^{\infty} \frac{1}{x^2+1}\,dx$ converges

$\Longrightarrow \sum \frac{1}{n^2+1}$ converges

(but not necessarily to $\pi/4$)

The P-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if } p>1$$

diverges if $p \le 1$

since $\int_1^{\infty} \frac{1}{x^p}\,dx = \frac{1}{1-p} x^{1-p}\Big|_1^{\infty} = \frac{1}{1-p}\left[\lim_{a\to\infty} a^{1-p} - 1\right]$

if $p>1$ then $1-p<0 \Rightarrow a^{1-p} = \frac{1}{a^{|1-p|}} < a$

if $p \le 1$ then $1-p \ge 0 \Rightarrow a^{1-p} > a$

Comparison Test

If $\sum b_n$ is convergent and $a_n \leq b_n$ $\forall n$ then $\sum a_n$ is also convergent.
If $\sum b_n$ is divergent and $a_n \geq b_n$ $\forall n$ then $\sum a_n$ is also divergent.

Of course, to use this we need a good b_n.
we usually use
- A p-series [$\sum \frac{1}{n^p}$ converges if $p>1$, diverges $p\leq 1$]
- Geometric Series [$\sum ar^{n-1}$ converges $|r|<1$ diverges if $|r|\geq 1$]
- harmonic series [$\sum \frac{1}{n}$ diverges]

ex. does $\sum \frac{1}{2+3^n}$ converge or diverge

$\frac{1}{2+3^n} \leq \frac{1}{3^n}$ which is geometric w/ $a=1$ $r=\frac{1}{3}$ and converges

$\Rightarrow \sum \frac{1}{2+3^n}$ converges

ex. $\sum \frac{5}{n}$? $\frac{5}{n} \geq \frac{1}{n} \leftarrow$ harmonic diverges

So $\sum \frac{5}{n}$ diverges

Limit Comparison Test

If $\lim_{n\to\infty} \frac{a_n}{b_n} = C$ where c is finite # & $c>0$ then either both series converge or both series diverge.

How we use this:

I ask does $\sum S_n$ div? or con?

you choose $S_n = a_n$ or $S_n = b_n$

then choose other (a_n or b_n) to be some series that makes $\lim_{n\to\infty} \frac{a_n}{b_n} = C$ & we know if it converges or diverges?

ex) Does $\sum \frac{\sqrt{n}}{n^2+1}$ Converge? Div?

let $a_n = \frac{\sqrt{n}}{n^2+1}$ $b_n = \frac{1}{n^{3/2}}$ ← convergent p-series

then $\lim_{n\to\infty} \frac{\sqrt{n}/n^2+1}{1/n^{3/2}} = \lim_{n\to\infty} \frac{n^{1/2} n^{3/2}}{n^2+1}$

$= \lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{1}{1+1/n^2} = \frac{1}{1+0} = 1$

So $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ and b_n converges $\Rightarrow a_n$ Converges

$\therefore \sum \frac{\sqrt{n}}{n^2+1}$ converges