8.4

Note Title 3/17/2008

<u>Alternating Series Test</u> (alternate + and −)

If the alternating series
$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$$
satisfies
i) $b_{n+1} \leq b_n \quad \forall n$
ii) $\lim_{n\to\infty} b_n = 0$
then the series converges.

Says "if absolute values of terms decrease to zero then series converges"

a. $\sum \frac{n}{(-2)^{n-1}} = \sum (-1)^{n-1} \frac{n}{2^{n-1}}$ since $(-2)^{n-1} = (-1)^{n-1}(2)^{n-1}$

let $b_n = \frac{n}{2^{n-1}}$
we need to show that b_n is decreasing

$$\frac{n}{2^{n-1}} \overset{?}{\geq} \frac{n+1}{2^{(n+1)-1}} = \frac{n+1}{2^n}$$
$$n2^n \overset{?}{\geq} (n+1)2^{n-1}$$
$$n2^n \cdot 2 \overset{?}{\geq} (n+1)2^{n-1} \cdot 2$$
$$2n2^n \overset{?}{\geq} (n+1)2^n$$
$$2n \overset{?}{\geq} n+1$$

$2n \geq n+1$ when $n \geq 1 \Rightarrow b_n \geq b_{n+1}$
$\therefore$ b_n is decreasing

now need to show that $\lim_{n\to\infty} b_n = 0$

$$\lim_{n\to\infty} \frac{n}{2^{n-1}} \overset{H}{=} \lim_{n\to\infty} \frac{1}{2^{n-1}\ln 2} = 0$$

by alternating series test

$\sum \frac{n}{(-2)^{n-1}}$ converges

Absolute Convergence

A series $\sum a_n$ is absolutely convergent if $\sum |a_n|$ is convergent.

Thm.

If a series is absolutely convergent then it is also convergent.

ex. $\sum (-1)^n \frac{n}{n^2+1}$ is absolutely convergent?

Let $a_n = \frac{n}{n^2+1}$ and $b_n = 1/n$ ← note: harmonic div.

$$\lim_{n\to\infty} \frac{\frac{n}{n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{1}{1+1/n^2} = 1$$

by Limit Comparison Test a_n diverges

Ratio Test

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$ then the series $\sum a_n$ is absolutely convergent ($\therefore$ convergent)

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$ then the series $\sum a_n$ is divergent

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the ratio test is inconclusive. (can't say.)

ex. $\sum \frac{n^2 2^{n+1}}{3^n}$ conv or div?

$$\lim_{n\to\infty}\left|\frac{\frac{(n+1)^2 2^{n+1+1}}{3^{n+1}}}{\frac{n^2 2^{n+1}}{3^n}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2 \cancel{2^n} 2^{\cancel{2}} \cancel{3^n}}{\cancel{3^n} 3 n^2 \cancel{2^n} \cdot \cancel{2}}\right|$$

$$= \frac{2}{3}\lim_{n\to\infty}\left|\frac{(n+1)^2}{n^2}\right| = \frac{2}{3}\lim_{n\to\infty}\frac{n^2+2n+1}{n^2}$$

$$= \frac{2}{3}\lim_{n\to\infty}\frac{1+2/n+1/n^2}{1} = \frac{2}{3}\cdot 1 = 2/3 < 1$$

so converges.