8.7

Note Title 3/18/2008

Taylor's Series

Let

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

$$|x-a| < R$$

$$\Rightarrow f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

$$f''(x) = 2c_2 + 2\cdot 3c_3(x-a) + 3\cdot 4c_4(x-a)^2 + \cdots$$

$$f'''(x) = 2\cdot 3c_3 + 2\cdot 3\cdot 4(x-a) + 3\cdot 4\cdot 5c_5(x-a)^2 + \cdots$$

$$\vdots$$

If we let $x = a$ then $f'(a) = c_1$

$$f''(a) = 2c_2$$

$$f''(a) = 2\cdot 3c_3$$

$$\vdots$$

$$f^{(n)}(a) = n!\cdot c_n$$

Solve this for c_n we get

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Thm

If f has a power series expansion of

$$f(x) = \sum c_n(x-a)^n \qquad |x-a| < R$$

then its coefficients are

$$c_n = \frac{f^{(n)}(a)}{n!}$$

This gives The Taylor Series of f at a.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

A special case of the Taylor Series is when $a=0$ then we get

$$f(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

this special case is called the <u>Maclaurin series</u>

ex. find Taylor Series for $f(x) = 1/x$ centered at 1. Use this to find $f(1/2)$.

$f(x) = 1/x = x^{-1}$ $\qquad$ $f(1) = 1$

$f'(x) = -x^{-2}$ $\qquad$ $f'(1) = -1$

$f''(x) = 2x^{-3}$ $\qquad$ $f''(1) = 2$

$f'''(x) = -6x^{-4} = -2\cdot 3x^{-4}$ $\qquad$ $f'''(1) = -6\cdot 1$

$f^{(4)}(x) = 24x^{-5} = 2\cdot 3\cdot 4x^{-5}$ $\qquad$ $f^{(4)}(1) = 24\cdot 1$

$f^{(5)}(x) = -120x^{-6} = -2\cdot 3\cdot 4\cdot 5x^{-6}$ $\qquad$ $f^{(5)}(1) = -120\cdot 1$

$\vdots$ $\qquad$ $\vdots$

$$f^{(n)}(1) = (-1)^n n!$$

So Taylor Series is

$$\frac{1}{x} = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \dots$$

$$= 1 - (x-1) + \frac{2}{2!}(x-1)^2 - \frac{6}{3!}(x-1)^3 + \dots$$

$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \dots$$

$$= \sum \frac{(-1)^n \cancel{n!}}{\cancel{n!}}(x-1)^n$$

$$f(x) = \sum (-1)^n (x-1)^n$$

To find $f(1/2)$

$$f(1/2) = \sum (-1)^n (1/2 - 1)^n = \sum (-1)^n (-1/2)^n$$

$$= \sum (1/2)^n$$

this is a geometric series with $a = 1$, $r = 1/2$

so

$$= \frac{1}{1 - 1/2} = \frac{1}{1/2} = 2$$ ✓

Remember Maclaurin series centered at 0.

Some important Maclaurin Series

(for proofs see book)

$$\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n \qquad e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$\sin x=\sum(-1)^n\frac{x^{2n+1}}{(2n+1)!} \qquad \cos x=\sum(-1)^n\frac{x^{2n}}{(2n)!}$$

$$\tan^{-1}x=\sum(-1)^n\frac{x^{2n+1}}{2n+1}$$

Useful fact!

$$\lim_{n\to\infty}\frac{x^n}{n!}=0 \quad \forall x\in\mathbb{R}$$

Multiplying/Dividing Power Series

Just like polynomials but we only care about the first few terms.

(ex) find first three nonzero terms of Maclaurin series of $y=e^x\frac{1}{1-x}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\Rightarrow e^x \frac{1}{1-x} = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)\left(1 + x + x^2 + x^3 + \cdots\right)$$

$$\begin{array}{rllll}
 & 1 + x + x^2 + \cdots & & \leftarrow & 1 * (1 + x + x^2 + x^3 \cdots) \\
\longrightarrow & \quad x + x^2 + \cdots & & \leftarrow & x * (1 + x + x^2 + x^3 \cdots) \\
+ & \qquad \frac{1}{2}x^2 + \cdots & & \leftarrow & \frac{x^2}{2} * (1 + x + x^2 + x^3 \cdots) \\
\hline
 & 1 + 2x + 2.5x^2 + \cdots & & &
\end{array}$$

Ex. find Maclaurin series for e^{2x}

from above $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{2x} = \sum \frac{(2x)^n}{n!} = \sum \frac{2^n x^n}{n!}$