Note Title 2/13/2006

Section 3.5 Rational Functions and their Graphs

A <u>rational function</u> is a quotient of polynomial functions:

$f(x) = \frac{p(x)}{g(x)}$ where $p(x)$ and $g(x)$ are polynomial functions

Note: we must have $g(x) \neq 0$

<u>Examples</u>: Rational functions

ⓐ $f(x) = \frac{x^3+2x+1}{x^2-6x}$

ⓑ $f(x) = \frac{x^2-16}{x+4}$

<u>Domain</u> of rational functions: All real numbers except those that make the denominator 0

<u>Example</u>: ⓐ above: $f(x) = \frac{x^3+2x+1}{x^2-6x}$

Find domain: all x values except where $x^2-6x=0$

$x(x-6)=0$

$x=0$ or $x=6$

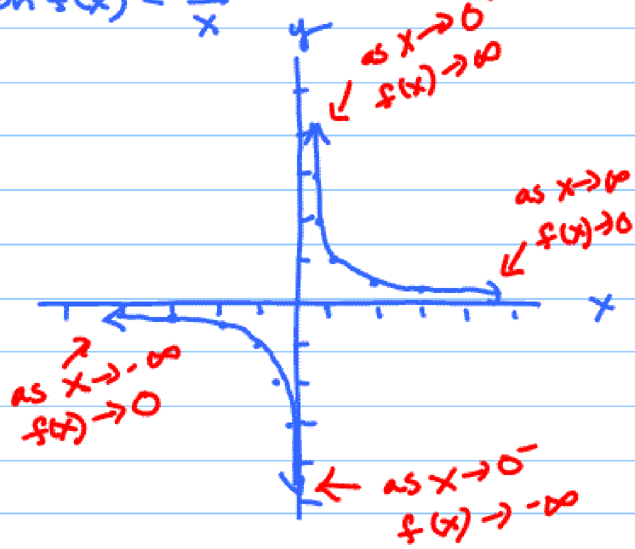
domain: $\{x \mid x \neq 0, x \neq 6\}$

Another <u>friendly familiar function</u>:

<u>Reciprocal function</u>: $f(x) = \frac{1}{x}$

Ex: Graph $f(x) = \frac{1}{x}$

x	y
1	1
2	1/2
3	1/3
1/2	2
1/3	3
-1	-1
-2	-1/2
-3	-1/3
-1/2	-2
-1/3	-3



Notation: As $x \to 0^+$, $f(x) \to \infty$

"As x approaches (gets close to) 0 from the right f(x) approaches (gets close to) infinity."

As $x \to 0^-$, $f(x) \to -\infty$

As $x \to \infty$, $f(x) \to 0$

As $x \to -\infty$, $f(x) \to 0$

Notes: 1. the graph gets close to but never crosses the x-axis ($y=0$). we say the x-axis (the line $y=0$) is a horizontal asymptote.

2. the graph gets close to but never crosses the y-axis ($x=0$). we say the y-axis (the line $x=0$) is a vertical asymptote.

Vertical asymptotes

we say the line $x=a$ is a <u>vertical asymptote</u> if $f(x)$ approaches $+\infty$ or $-\infty$ as x approaches a

<u>Notation</u>: $f(x) \to \pm\infty$ as $x \to a$

<u>Translation</u>: $f(x)$ gets really big or really small for x close to a

<u>Notes</u>: 1. $f(x)$ can have several vert. asympts.
2. $f(x)$ will not cross its vert. asympts.

Finding vertical asymptotes

Given a rational function
$f(x) = \frac{p(x)}{q(x)}$ in which $p(x)$ and $q(x)$ have <u>no common factors</u>

If $q(a)=0$, then $f(x)$ will have a vertical asymptote at $x=a$

<u>Translation</u>: Find those values of x that make the denom. $= 0$. $f(x)$ will have vertical asymptotes there

ex $f(x) = \frac{2x+4}{x^2+3x+2}$

$$= \frac{2\cancel{(x+2)}}{\cancel{(x+2)}(x+1)} = \frac{2}{x+1} \leftarrow x+1=0 \text{ when } x=-1$$

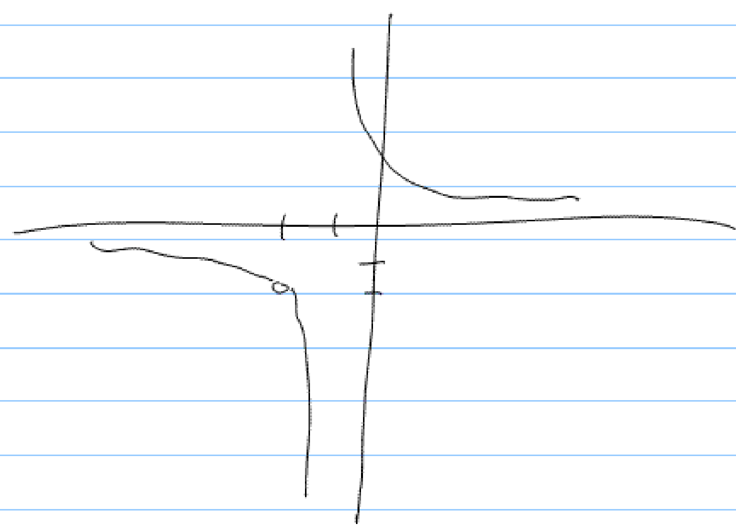
So there's a vertical asymptote at $x=-1$.

→ check w/ graphing calculator.

notice there is NOT a V.A. at $x=-2$ However $f(x)$ is undefined there (since $f(-2)=\frac{2(-2)+4}{(-2)^2+3(-2)+2}=\frac{-4+4}{4-6+2}=\frac{0}{0}$)

this means there is a <u>hole</u> at $x=-2$.

Our calculators do NOT show us this.



<u>Horizontal Asymptotes</u>

We saw that we had a horiz. asymptote at $y=0$ when we graphed $f(x)=\frac{1}{x}$

we say that the line $y=b$ is a <u>horizontal asymptote</u> if $f(x)$ approaches b as x approaches $+\infty$ or $-\infty$.

<u>Notation</u>: $f(x) \to b$ as $x \to \pm\infty$

<u>Translation</u>: $f(x)$ gets close to b as x gets really big or really small

<u>Notes</u>: 1. $f(x)$ can have <u>at most one</u> horizontal asymptote

2. $f(x)$ may cross its horiz. asympt.

<u>Finding horizontal asymptotes</u>:

Given a rational function

$f(x) = \frac{p(x)}{g(x)}$, where degree of $p(x) = n$, leading coeff. $= a_n$

degree of $g(x) = m$, leading coeff. $= a_m$

① If $n < m$, then the x-axis ($y=0$) is the horizontal asymptote.

② If $n = m$, the line $y = \frac{a_n}{a_m}$ is the horiz. asympt.

③ If $n > m$, the graph of f has no horizontal asymptotes.

aka

1. If degree(numerator) < degree(denominator) then $y=0$ is HA

2. If degrees are equal then

$$y = \frac{\text{leading coefficient of num.}}{\text{leading coefficient of den.}}$$

3. If degree(numerator) > degree(denominator) then there's no HA.

ex) $f(x) = \dfrac{x-2}{x^2+8}$

since degree of num. = 1 and
degree of den = 2
the line $y=0$ is the HA
(x-axis)

ex) $g(x) = \dfrac{8x^4+7}{4x^4-3x^2+10}$

since degree of num = 4 and
degree of den = 4
the line
$y = \frac{8}{4}$ (or $y=2$) is HA

ex) $h(x) = \dfrac{x^4+7}{x^2+8}$

since degree of num = 4
degree of den = 2
there is no HA.

Putting it all together:

Strategy for Graphing a Rational Function:

Given a rational function $f(x) = \frac{p(x)}{q(x)}$ in which $p(x)$ and $q(x)$ have no common factors.

1. Find y-intercept by finding $f(0)$
2. Find x-intercepts by solving $p(x) = 0$
3. Find vertical asymptotes by solving $q(x) = 0$
4. Find horizontal asymptote by comparing degrees of $p(x)$ and $q(x)$
5. Look at possible symmetry:
   $f(-x) = f(x)$ y-axis symmetry
   $f(-x) = -f(x)$ origin symmetry
6. Plot at least one point between and beyond each x-intercept and vertical asymptote
7. Plot a few more points to verify what you have found

Or use graphing calculator!

Note: Don't worry about slant asymptotes yet ex $f(x) = \dfrac{x+5}{x^2-3x+1}$

1. $f(0) = 5$ y-intercept $= 5$
2. $f(x) = 0 \Rightarrow x = -5$ x-intercept $= -5$
3. $x^2 - 3x + 1 = 0 \Rightarrow$

   $x = \frac{3}{2} \pm \frac{\sqrt{5}}{2} \approx 2.618\ldots$ and $.3819$

4. deg of num. = 1, deg of den. = 2 $\Big\}$ so $y=0$ is HA

5-8. look at calculator for general shape.