Section 4.1 Exponential Functions

Note Title 2/21/2006

We've seen expressions such as

$2^3, 2^{1/2}, 2^{-7}$, etc.

what if we consider the expression

$2^x \leftarrow$ exponent is a variable

consider the equation

$$y = \underset{\substack{\uparrow \\ \text{base} = 2}}{2}^{x}$$

This equation is a <u>function</u>

we could also have a different base:

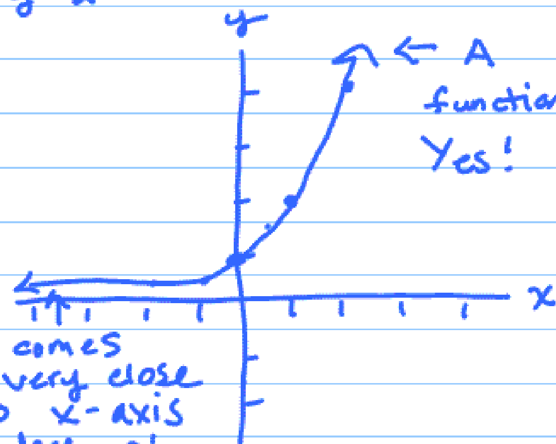
<u>Ex</u>: $\left.\begin{aligned} f(x) &= 5^x \\ f(x) &= 3^x \\ f(x) &= \left(\tfrac{1}{2}\right)^x \end{aligned}\right\}$ exponential functions

<u>Definition:</u> A function of the form $f(x) = b^x$ is an <u>exponential function with base b</u> where $b > 0, b \neq 1$, and x is a real number

Ex.: Graph $f(x)=2^x$

$y=2^x$

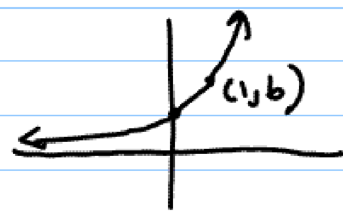
x	y
0	1
1	2
2	4
-1	$2^{-1}=\frac{1}{2}$
-2	$2^{-2}=\frac{1}{4}$
$\frac{1}{2}$	$2^{1/2}=\sqrt{2}$



← A function? Yes!

comes very close to x-axis but does not cross it because $f(x)=2^x$ is never $=0$

For $b>1$, $f(x)=b^x$ will have same basic shape as above:



$f(x)=b^x$

$b>1$

General characteristics of graph of $f(x)=b^x$:

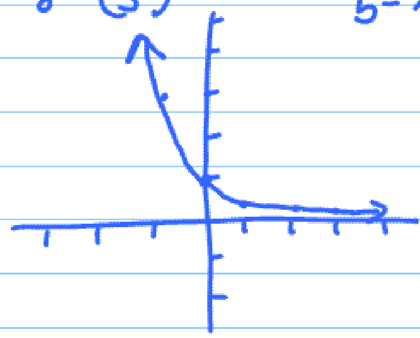
1. y-intercept $=(0,1)$
2. Passes through $(1,b)$

3. Increases from left to right for $b>1$
   Decreases from l to r for $0<b<1$
4. No <u>x-intercepts</u> (approaches x-axis but never crosses)
5. Domain: $\{x \mid x \text{ is a real number}\}$
6. Range: $\{y \mid y>0\}$

<u>Ex</u>.: Graph $f(x)=\left(\frac{1}{3}\right)^x$

$y=\left(\frac{1}{3}\right)^x$ $b=1/3$

x	y
0	1
1	1/3
2	1/9
3	1/27
-1	$(1/3)^{-1}=3$
-2	$(1/3)^{-2}=9$



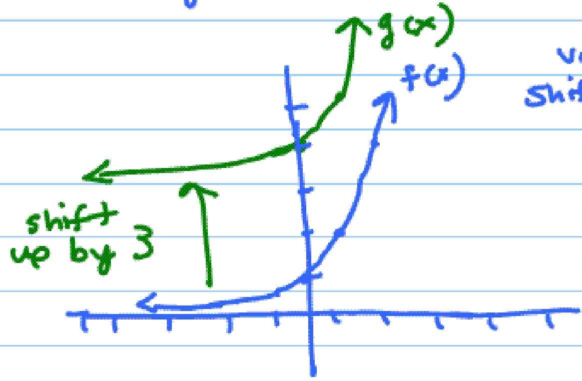
So for $0<b<1$, $f(x)=b^x$ has same basic shape as above example

Can use <u>transformations</u> to graph exponential functions (same principles as in Section 2.5 - use same table - pg. 250)

Ex: Graph $g(x) = 2^x + 3$
compare to $f(x) = 2^x$

$g(x) = 2^x + 3 = f(x) + 3$

vertical shift up by 3



The Natural Base: e

In real-life, one particular base occurs frequently. It happens to be an irrational number (non-repeating, non-terminating) which we call $\underline{\underline{e}}$.

$e \approx 2.71828\ldots$

What is e? Consider the expression

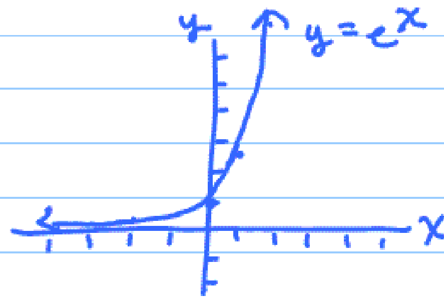
$\left(1+\frac{1}{n}\right)^n$

n	$\left(1+\frac{1}{n}\right)^n$
1	2
5	2.48832
500	2.71556...
50,000	2.71825...

As n gets very large, $\left(1+\frac{1}{n}\right)^n$ get close to e:
As $n \to \infty$, $\left(1+\frac{1}{n}\right)^n \to e$

Ex: Graph $f(x) = e^x$ (use calculator)

x	y
0	1
1	≈ 2.7
2	≈ 7.4
-1	$1/e \approx .4$
-2	$1/e^2 \approx .1$



Applications

The number e occurs in many real-life applications such as calculating population growth and compound interest

Compound Interest

Suppose you invest a certain amount of money, called the principal, P, in an account at annual percentage rate, r, compounded annually.

After one year, the amount, A_1, in your account will be

$$A_1 = \underbrace{P}_{\text{original amount invested}} + \underbrace{Pr}_{\text{interest}} = P(1+r)$$

After two years:

$$A_2 = A_1 + A_1 r = A_1(1+r)$$
$$= P(1+r)(1+r)$$
$$= P(1+r)^2$$

In general we have:

# years	amount in account
0	$A=P$
1	$A=P(1+r)$
2	$A=P(1+r)^2$
3	$A=P(1+r)^3$
⋮	
t	$A=P(1+r)^t$

So, $\boxed{A = P(1+r)^t}$ gives the balance in an account with principal P, at interest rate r, for t years

Now, suppose we want to allow more frequent compounding (monthly, daily, etc.)

Let n = # compoundings per year

Then $\frac{r}{n}$ = interest rate per compounding

In general:

After t years with n compounding periods per year, the amount in the account (balance) is given by

$$\boxed{A = P\left(1+\frac{r}{n}\right)^{nt}}$$

If we let the # compoundings increase (so $n \to \infty$), we get <u>continuous compounding</u>.

For continuous compounding, the balance is given by

$$\boxed{A = Pe^{rt}}$$

<u>Note</u>: You should know how to <u>use</u> these formulas. See Example 6, pg. 381

ex.) You invest \$1000 in an account earning 8% for 10 years. How much \$ do you have if interest is compounded monthly? Continuously?

monthly

Formula: $A = P\left(1+\frac{r}{n}\right)^{nt}$

$P = \$1000 \quad r = .08 \quad n = 12 \quad t = 10$

$A = 1000\left(1+\frac{.08}{12}\right)^{12 \cdot 10} \approx \$2219.64$

Continuously

Formula: $A = Pe^{rt}$

$P = \$1000 \quad r = .08 \quad t = 10$

$A = 1000e^{.08 \cdot 10} \approx 2225.540928\ldots$

$\approx \$2225.54$