Section 8.5 The Binomial Theorem

Note Title 4/11/2006

Recall: A binomial is a polynomial with two terms

consider the expression $(x+y)^n$

We already know

$$(x+y)^0 = 1$$
$$(x+y)^1 = x+y$$
$$(x+y)^2 = x^2+2xy+y^2$$
$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$

can we find a formula for $(x+y)^n$?

Note: $(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$

$(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$

Do you see a pattern for powers of variables?

Observations:

1. The powers of x decrease by 1 in successive terms and the powers of y increase by 1
2. The sum of the powers in each term is n
3. The number of terms is n+1
4. The first term is x^n, the last term is y^n

Example: Find the variable parts of $(x+y)^6$

Solution: $x^6 + \_\_x^5y + \_\_x^4y^2 + \_\_x^3y^3 + \_\_x^2y^4 + \_\_xy^5 + y^6$

What about the coefficients?
If we write only the coefficients of the expressions from the above expansions we have:

Coefficients for

$(x+y)^0$	1
$(x+y)^1$	1 1
$(x+y)^2$	1 2 1
$(x+y)^3$	1 3 3 1
$(x+y)^4$	1 4 6 4 1
$(x+y)^5$	1 5 10 10 5 1
$(x+y)^6$? →	1 6 15 20 15 6 1

Pascal's Triangle

Notes: ① First and last terms of each row are 1

② other numbers are from the sum of the two above it

[Some other cool things about Pascal's Triangle:

1. sum of rows ⇒ powers of 2

   Example: $2^5 = 1+5+10+10+5+1 = 32$

2. powers of 11 ⇒ read across a row

   Example: $11^2 \Rightarrow 121$

3. can find Fibonacci numbers along the diagonals]

Example: write expansion for $(x+y)^6$

$$(x+y)^6 = \underline{\ }x^6 + \underline{6}x^5y + \underline{15}x^4y^2 + \underline{20}x^3y^3 + \underline{15}x^2y^4 + \underline{6}xy^5 + \underline{\ }y^6$$

Another way for finding coefficients of a binomial $(x+y)^n$ — useful for large powers of n:

First we need some notation:

$$\underbrace{{}_nC_r = \binom{n}{r}}_{\text{"n choose r"}} = \frac{n!}{r!(n-r)!}$$

Examples: Find the following:

ⓐ $\binom{7}{3} = \frac{7!}{3!(7-3)!} = \frac{7!}{3!\,4!} = \frac{7\cdot6\cdot5\cdot\cancel{4!}}{3\cdot2\cdot1\cdot\cancel{4!}} = 35$

ⓑ $\binom{4}{0} = \frac{4!}{0!(4-0)!} = \frac{\cancel{4!}}{0!\,\cancel{4!}} = \frac{1}{1} = 1$

Putting it all together we have

* The Binomial Theorem

For any positive integer n,

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \dots + \binom{n}{n}b^n$$

This is easier to use than it looks

Helpful hint: $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$

Example: Use the Binomial Theorem to find $(x+2)^5$

Solution:

$$(x+2)^5 = \binom{5}{0}x^5 + \binom{5}{1}x^4 \cdot 2 + \binom{5}{2}x^3 \cdot 2^2 + \binom{5}{3}x^2 \cdot 2^3 + \binom{5}{4}x \cdot 2^4 + \binom{5}{5}2^5$$

$$= 1 \cdot x^5 + \frac{5!}{1!4!}2x^4 + \frac{5!}{2!3!}4x^3 + \frac{5!}{3!2!}8x^2 + \frac{5!}{4!1!}16x + 1 \cdot 32$$

$$= x^5 + 5 \cdot 2x^4 + 10 \cdot 4x^3 + 10 \cdot 8x^2 + 5 \cdot 16x + 32$$

$$= x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$$

Find a particular term of a Binomial Expansion

Suppose we want to find a specific term of $(a+b)^n$

Example: Find the fourth term of $(x+2)^5$

(Note: the fourth term was found above as $\binom{5}{3}x^2 2^3$)

* $\boxed{\text{In general, the } r\text{th term of } (a+b)^n \text{ is: } \binom{n}{r-1} a^{n-(r-1)} b^{r-1}}$

Note: this formula is slightly different from the one in the book

Example: Find the sixth term in $(x^3-y^2)^8$

$n=8$, $r=6$, $a=x^3$, $b=-y^2$

so, 6th term: $\binom{8}{5}(x^3)^3(-y^2)^5$

$= \frac{8!}{5!\,3!} x^9(-y^{10})$

$= \frac{8\cdot 7\cdot 6\cdot \cancel{5!}}{\cancel{5!}\ 3\cdot 2\cdot 1} x^9(-y^{10})$

$= 56x^9(-y^{10})$

$= -56x^9y^{10}$